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Why Johnny hates math—

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Many difficulties encountered in learning mathematics are caused by a confused system of symbols and terminology

THE PROBLEM

"MATHEMATICS the most-hated Subject" was the title of an editorial that recently appeared in a large newspaper. "According to a national survey of high school seniors," says a report sponsored by the Carnegie Foundation, "twelve per cent of them had never taken any algebra or geometry; twenty-six per cent had quit studying mathematics after one year; and another thirty per cent had dropped the subject by the end of the second year." In view of the unprecedented national need for scientists with an adequate basic training in mathematics "there is cause for great alarm," and "concern over the mathematical incompetence of the average—and even above-average—American has become almost a national preoccupation." All associations and foundations connected with either science or education have expressed grave warnings, and large p blic as well as private funds go into projects whose aims are to explain why mathematics is so unpopular and to alleviate the resulting crisis.

Discussion has brought forth a great variety of partial explanations. Experts have blamed the teachers and the students. They have criticized the universities, the teachers colleges, the high schools, and the grade schools. They have found fault with the choice of classical topics, their arrangement, and the absence of modern mathematics from the curriculum. They have emphasized that the current textbooks are obsolete, the classrooms overcrowded, and the teachers underpaid. They have, in other words, searched for the roots of the trouble practically everywhere—everywhere except in the procedures of mathematics itself.

Mathematics is looked upon with a mixture of awe and gratitude. And indeed everyone has reason to be grateful. Mathematics has been the decisive factor in understanding the universe and the most powerful tool in controlling nature. All this is beyond any doubt. What is on shakier ground, however, is the general belief that the tremendous achievements of mathematics are due to the clarity of its basic procedures and to the precision of its current language. The time has come when it must be frankly admitted that some mathematical processes are successful not because of those presumed qualities, but rather in spite of obscure foundations, ambiguous expressions, and lack of articulateness. None of these shortcomings—it goes without saying—are in any way bothersome to active mathematicians, who handle their material with a virtuosity that is based on intuition and experience. Mathematicians have the skill to cope with self-created difficulties. But Johnny, studying algebra in high school or college, lacks that skill and is unable to understand part of what he hears from his teacher or reads in the textbook, and this is the reason why Johnny hates math.

THE FAULT IS NOT THE STUDENT'S

The mathematical symbols that the student encounters most frequently—on almost every line on every page of his texts—are letters, especially the letter x. Yet he is rarely taught clear and articulate directions for the use of letters. In the formula

(1) $x^2-1=(x+1)(x-1)$

he may replace x, for instance, with 4 or with 7, since $4^2-1=(4+1)\ (4-1)$ and $7^2-1=(7+1)\ (7-1)$.

But shame on Johnny if he replaces x with 4 or 7 in some of the other statements that he finds in his text, e.g., in

(2) the function x^2-1 is nonconstant

or in

(3) x^2-1 is a quadratic polynomial.

For 4^2-1 and 7^2-1 are called neither nonconstant functions nor quadratic polynomials. Johnny, however, resents being blamed for doing in (2) and (3) precisely what he had been expected to do in (1). Very understandably, he thinks that the game is unfair because he has not been told the rules. Indeed, where he may, and where he must not, replace letters by numerals is something the student has to find out himself. This is a difficult task.

Nor are the diverse uses of x in (1), (2), and (3) the only ones that the beginner encounters. In the elements of algebra he has to find the number x such that

(4) $x+1=5.$

This task is called solving an equation. The student is expected to say, "The solution is $x=4$."

In the elements of analytic geometry (usually incorporated in elementary algebra), a Cartesian frame of reference is chosen and a simple procedure taught according to which the line parallel to the second axis and 4 units to its right is called

(5) the line $x=4$.

Thereafter, the student is expected to refer to $x=4$ as the equation of the said straight line. This "equation" is worlds apart from the equation (4) or its solution, $x=4$, and so is the letter x in (5) from the letter x in (4). Furthermore, both are different from the letters x in (1), (2), and (3).

It can be pointed out that physicists also use the same letter in different meanings. They write t for *time* as well as for *temperature*, for example. But they do so in different contexts, and no student confuses the two meanings of t. However, the student is bound to confuse various mathematical meanings of x because these often occur in the same context and sometimes even in the same sentence. For instance, in his study of the line $x=4$, the student may well have to solve equations such as (4) with the solution $x=4$. Formulas such as (1) may well occur in the same context as statements (2) and (3), even though they are based on totally different uses of x. Mathematics suffers from x-itis.

The task is not made easier by teachers and texts referring to the same letter x by different technical terms. It is a *variable* in (1) and sometimes also in (2); while x in (2) and (3) is sometimes called a *function*. In other contexts x is called an *indeterminate* in (3). It is called an *unknown* in (4) and an *abscissa* in (5). Slightly more advanced parts of algebra and analytic geometry abound in references to x as *real part*, as *parameter*, and again as a *function*—but a function quite different from that in (2) and sometimes called the function x of x and y. The survivor of freshman mathematics finds that x in $\int_0^1 \cos x\,dx$ is called a *dummy* and, later, encounters references to the *operator* x. These distinctions are not very helpful since many students fail to grasp the meanings of those terms. Even though at least eight of the terms occur over and over and over again, some of them (especially indeterminate and parameter) are introduced without precise definitions,

even the differences between them (especially between *variables*, *parameters*, and *indeterminates*) are not clearly described, and articulate rules for the various ways to manipulate them (especially *variables*, *functions*, and *indeterminates*) are lacking.[1]

The student's task is made even harder if he is told[2] that the last letters of the alphabet are variables (standing for any one of many numbers), whereas the first letters of the alphabet are constants (each designating just one number). He cannot find any difference in meaning between the following two statements:

$x+1=1+x$ for every number x,

$a+1=1+a$ for every number a.

He cannot find any difference because there is none. The two statements have precisely the same meaning. "But how can that be?" he wonders, considering that the last letters of the alphabet are variables and the first letters are constants. He does not yet know that much worse inconsistencies are in store for him. After he has solved a few equations such as $x^2=4$ and $x^2=7$ he will be taught to solve the general equation

$$x^2=c, \tag{6}$$

from which the specific equations result by replacing c with 4 or 7. The letter c in (6) thus stands for any one of many numbers. Yet c is referred to as a constant and not as a variable.

If Johnny is to understand mathematics he has to learn quite a few oddities of the mathematicians' traditional jargon. Figuratively speaking, white is occasionally referred to as "black."

Nor is the teacher to blame

"Variable is perhaps the most mathematical of all notions," wrote Bertrand Russell, one of the great logicians of our time. "It certainly is also one of the most difficult to understand . . . In the present work [*The Principles of Mathematics*, 1903] a satisfactory theory as to its nature, in spite of much discussion, will hardly be found." The search for the solution of some of those difficulties is left to the high school and college teachers.

The last five decades have witnessed the rise of what is called modern mathematics. New branches of mathematical knowledge, undreamed of in 1903, have come into existence in a development so splendid that mathematicians have forgotten, or thought that they could ignore, those words in the *Principles*. Modern ideas make geometry much more lucid as well as much more interesting than it used to be. But just the difficulties Russell alluded to in 1903 have lingered on unabatedly. Mathematicians are eagles that have not freed, nor even tried to free, their legs from shackles that proved harmless in their upward flight. Johnny, however, does not fly. He is learning to walk and in this process, very understandably, he considers shackles on his legs a nuisance that he asks his teacher to remove. It appears, however, that, regardless of whether the teacher has seen the grandiose new vistas of modern mathematics, in some points he still cannot do better than Bertrand Russell did in 1903. Within the frame of the traditional mathematical symbolism he simply cannot answer some of the student's pertinent questions about variables and constants and about x and y.

Indeed, what shall the teacher say when the student (confusedly or clearly) asks in which cases x may, and in which cases x most not be, replaced by numerals? What shall he say when pressed for the difference between a variable and an indeterminate? How shall he reconcile the definition of a constant as a symbol that stands for just one number with the use of the word constant in reference to the letter c in the equation

$$y=x+c \tag{7}$$

[1] See K. Menger, *What Are x and y?* forthcoming in the Mathematical Gazette, where twelve different uses of x and y are listed.

[2] According to O. Frink, this is the conventional usage. See O. Frink, *Mathematical Reviews* (June, 1955) pp. 575–576. See also G. Polya, *How to Solve It* (Princeton: Princeton University Press, 1945), p. 125.

in analytic geometry? What shall he answer when asked whether or not c in (7) and c in, say, the equation

$$x=c \tag{8}$$

are the same constant? And, for that matter, what is the answer to the question whether x in (7) and x in (8) are the same variable? More generally, when are two constants equal and when are two variables equal?

The teacher's position is anything but enviable since these questions are anything but stupid. Complacently silencing the inquiring student is not a way out of the crisis: Johnny will quit math after one semester. Saying "as you will learn later" now, and "as you have learned before" later, is neither honest nor efficient: Johnny will drop mathematics at the end of the second semester. Ridiculing those questions is of course the worst response: that makes Johnny hate math for the rest of his life.

But the teacher is not to blame if he cannot answer these and dozens of similar questions. He himself has never been told the answers.

Where the trouble lies

The very reasons that exculpate the student and the teacher point to the true cause of the difficulties: the current symbolism and prevailing basic procedures of mathematics. These are creations of the seventeenth century, which, to quote again from Russell's *Principles*, "however fruitful of results, involved a diminution of logical precision and a loss in subtlety of distinction." This frame of mathematics has been left untouched for over a quarter of a millennium. Some parts of the foundation were frozen at about 1700 and have remained so ever since. They still satisfy most mathematicians—for the simple reason that becoming a mathematician is practically tantamount to getting thoroughly familiar with, and used to, that antiquated frame and since, as Whitehead once so aptly put it, there is "... a passionate attachment of some leaders of thought to the cycle of ideas within which they received their own mental stimulus at an impressionable age." But that all is not well is clearly evinced by the difficulties that beset students and teachers of mathematics.

What requires a thorough analysis are the processes by which, in spite of all difficulties, so many beginners do grasp the subject or at least acquire an efficient manipulative skill. The situation may perhaps be described as follows.

Children learn with amazing ease some rather complicated arithmetical conventions, e.g., that $10-3-2$ equals $7-2$ (and not $10-1$), whereas $10-3\times 2$ equals $10-6$ (and not 7×2). This and much more, if their teachers are successful, they learn in grade school. Not in explicit rules, though. Such rules would be entirely beyond a child's comprehension. Children learn the conventional use of minus and times signs by observing what the teacher writes and by guessing what he thinks while writing. And they begin to conform to the rules in tentative experiments and by imitating the teacher. This means that children learn arithmetical symbols just as, a few years earlier, they learned English words. They absorb conventions of arithmetic just as they absorbed the conventions of English grammar. The procedure is so thorough that both types of conventions become their second nature.

In high school and during their first year in college, adolescents are expected to learn algebra and analytic geometry essentially in the same manner. The terms (variable, indeterminate, parameter, etc.) are repeated over and over again, some of them without precise definitions or even clear descriptions of the differences between them. This combined with the lack of consistent rules for the use of x and y simply does not leave the student any alternative method of learning. And some students do learn algebra. In Johnny's view, some grasp the stuff because they

are geniuses, while others only pretend that they understand and make the teacher think they do so by their peculiar way of watching him do problems on the blackboard and then doing similar problems in tests. Johnny has a point there. Indeed, those who pass the course (including the geniuses) learn the use of those terms and symbols by observing what the teacher writes and by guessing what he thinks while writing. They begin to conform to the rules in tentative experiments and by imitating the teacher. Well, one might say, this should not be too hard for Johnny because it is precisely what he did as a child.

But the clairvoyance that made children painlessly learn their native tongue is blurred in many adolescents studying a foreign language. Acquiring more than a smattering proves to be very, very difficult and necessitates formal training in vocabulary and grammar. While the adult is superior to the child in reasoning and in understanding explicitly formulated rules, he is inferior in guessing the implicit and in instinctively grasping the intent of words. In the normal process of maturing, the logical faculties sharpen, but the devinatory powers blunt. As far as mathematics is concerned, arithmetic is something of a native tongue, whereas algebra, analytic geometry, and calculus today are foreign languages studied in adolescence. What Johnny really wants and (of course vaguely and inarticulately) asks for are, as it were, the vocabularies and grammars of these languages. But these the teacher cannot give him without, among other things, clarifying the differences (left unexplained in the current textbooks) between the basic concepts, refraining from occasional references to white as "black," and abandoning the indiscriminate use of x and y in totally diverse meanings. In other words, to satisfy Johnny's confusedly expressed but clearly justified demands, time-honored mathematical practices must be changed.

But why, one might ask, are matters coming to a head just in this country and just at this time?

The answer to the first question is that Johnny, who has a great deal of common sense and an incurable foible for calling white "white," has never enjoyed intellectual acrobatics as much as do his cousins abroad. Until recently no one cared. Now, all of a sudden, the nation desperately tries to sell him mathematics. On this buyer's market he dislikes the brand of mathematics offered to him and he balks. What may satisfy accomplished mathematicians is not good enough for Johnny. His naïveté, too often scoffed at, does not tolerate that "diminution of logical precision" mentioned by Russell. He is disgusted (to the point of dropping math) with what appears to him as a confusion beyond comprehension. Unconsciously he aims at regaining some of that "lost subtlety." Therefore, in the suit of Johnny vs. antiquated mathematical procedures, this writer wholeheartedly sides with the plaintiff.

That matters are coming to a head just in the 1950s is due to the postwar increase in the scale at which we disseminate (or at least attempt to disseminate) mathematico-scientific knowledge, thus, in a way, to the democratization of higher education.

Hence, some of the specific causes of the trouble just here and just now are gratifying indeed. Of course, this fact makes it even more imperative to solve the grave problem.

A PROPOSED SOLUTION

Simple questions about costs and return, which Johnny would rationally attack and readily answer on the street at any time, in mathematics classes often paralyze his brain. The idea of treating the problems in the context of algebra or calculus as he would treat them in any other connection does not even dawn upon him. For he regards mathematics as detached from, nay opposed to, common sense—that common sense of which he has more than his

share—instead of realizing that mathematics is the epitome of common sense. For the reasons indicated he mistakes mathematics as an abracadabra and tries to guess the right incantation instead of approaching algebra rationally.

Under these circumstances, the presentation of mathematics must be absolutely and impeccably rational. Even the slightest inconsistency in practices or terms is intolerable. Any contradiction, no matter how trivial to the initiated, only serves to strengthen some beginner's misconceptions about magic. The solution of the grave problem created by this misunderstanding of mathematics is the expurgation of everything even remotely resembling nonsense followed by an appeal to the students' common sense.

Various attempts are being made at the present time to revise the presentation of the classical system of mathematics. Only experiments on a large scale can decide merits and result in improvements of any particular approach. But it may perhaps be mentioned that students exposed to this writer's clarified treatment of the basic concepts and to his articulate rules for the use of the basic symbols eagerly absorb this vocabulary and this grammar of the mathematical languages and respond to this new approach enthusiastically. The essence of this treatment is summarized in these points:

1. A clear distinction is maintained between numerical variables such as x in (1) and variable quantities such as the abscissa in analytic geometry, e.g., in (4). Naturally, the same symbols are not used for variables and variable quantities. For instance, one may use roman type for the numerical variable and italic for the abscissa as in

(1*) $\mathrm{x}^2-1=(\mathrm{x}+1)(\mathrm{x}-1)$ for any number x

(4*) $x=4$.

This distinction can easily be maintained on blackboards and on paper by writing

$$\mathrm{x},\ \mathrm{y},\ \mathrm{z}$$

for variables and

$$\mathfrak{x},\ \mathfrak{y},\ \mathfrak{z}$$

for variable quantities.

2. A clear distinction is maintained between the numerical variable x in (1*), on the one hand, and the identity function, e.g., in (2), on the other. The identity function is defined by the process of associating with any number that very same number. The result of this process, the function assuming for any number x the value x, may therefore be considered as the class of all pairs (x, x) for any number x. While in this definition x is a variable, there is nothing variable about the identity function itself. Of course, the function is denoted by a symbol, say j, different from those used as numerical variables.

3. A clear distinction is maintained between a numerical variable such as x in the statement

$$\frac{\mathrm{x}^2-4}{\mathrm{x}-2}=\mathrm{x}+2 \text{ for any number } \mathrm{x}\neq 2,$$

and an indeterminate. The variable is meant to be replaced by the designations of specific numbers (in fact, of any number $\neq 2$), whereas an indeterminate is not. An indeterminate is rather of the nature of the imaginary i in

$$\frac{\mathrm{i}^2-4}{\mathrm{i}-2}=\mathrm{i}+2.$$

Here, i is not meant to be replaced by anything, wherefore no one would dream of qualifying the last formula by saying "provided $\mathrm{i}\neq 2$."

For numerous details of this new approach, such as the definitions of variable quantities, indeterminates, and parameters, the treatment of constants, etc., the reader must be referred to the author's books.[3]

[3] Karl Menger, *The Basic Concepts of Mathematics: A Companion to Current Textbooks.* (Mimeographed) (Chicago: Illinois Institute of Technology Bookstore, 1956), and Karl Menger, *Calculus: a Modern Approach* (Boston: Ginn and Company, 1955).

In the long run, a teacher will be successful only if in the spirit of Russell's humble admission of imperfection he deepens his insight into mathematics to the point where, perhaps contrary to what he may have learned or believed, he realizes that the material he has to teach urgently needs clarification. No one should be surprised that in unfreezing the 250-year-old foundation, the introduction of two or three new symbols becomes indispensable. What is surprising, and in fact a new triumph of Renaissance mathematics, is that overhauling its colossal machinery calls for so few and so small adjustments to bring it up-to-date. It is as though in building a skyscraper instead of a seventeenth-century château all one needs are a few extra hammers.

Getting accustomed to, and initiating the beginner into, the use of dissimilar symbols for

(a) variables,
(b) co-ordinates,
(c) the identity function,
(d) indeterminates,

and he has to initiate the beginner into this distinction. Since its maintenance is next to impossible within the frame of the current theory—that pan-x-ology in which all the diverse concepts (a)–(d) and others are denoted by x—one must denote unlike concepts by unlike symbols.

This is a modest price for the resulting clarification. It is this clarification that enables the teacher to answer dozens of otherwise highly embarrassing questions. The greatest benefit of the new approach is inherent. Since most of today's unanswerable questions concern the equivocal use of basic terms and symbols, with the new approach these questions cannot possibly arise.

Is a new approach worth the teacher's while?

The main obstacle to tryouts along basically new lines is a reasoning, apparently prevailing among some research mathematicians. It might be summarized as follows, "Since for the past two hundred years and to this day, all mathematicians and scientists have achieved complete mastery of mathematics with its time-honored procedures and in its traditional presentation, and since furthermore, the difficulties here discussed do not disturb any accomplished mathematician in the least, youngsters who cannot cope with them must be mathematically utterly incompetent. To revise procedures or symbols for their sake is not worth anyone's while since their study of mathematics cannot, under any circumstances, be profitable either to those mathematical morons themselves or to anyone else."

But this reasoning is a dangerous fallacy. For how can one rule out the possibility that some quite intelligent youngsters, deterred by incomprehensible basic material, have gone into other studies or occupations and thus never reach the mathematical level on which research men train talent? How can one even rule out the supposition that some logical minds are dissatisfied with mathematics as presented to them and abandon it at an early stage just because of their logical abilities? And how can this country with its need for trained mathematicians and scientists afford to go along with ideas that risk the loss of talent, perhaps even of especially valuable talent?

As a matter of fact, statistics seem to indicate that this loss of talent is very real. The national survey mentioned at the beginning of this paper shows that among the brighter high school seniors alone (that is, the top thirty per cent on a test of mental ability) more than forty per cent never get beyond the elementary phases of mathematics. And, passing from the statistics of masses to the individual, this writer will never forget the remark of an eminent legal authority, renowned for the acumen of his analytic mind, when he recalled "Mathematics is a field in which I wasn't any good in school. I never understood what x meant."

Still more on the cutting of squares

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Discussions of this subject in THE MATHEMATICS TEACHER *for May 1956 and October 1956 are here brought to culmination in a constructive proof of the theorem that any convex polygon is equivalent to a square. The present paper also provides material that supplements material in the same vein in* The Mathematics Student Journal *for December 1956.*

OUR FIRST and second papers[1] on the theme of cutting squares dealt with particular instances of the general problem:

> *Given a figure (or several figures), to cut same into parts and rearrange these parts in such a way that there results another figure (or figures) of some specified sort.*

The parts envisaged here are finite in number and are polygonal, i.e., have boundaries which are straight segments. To cut a figure into such parts and rearrange same into another is *to transform* the first figure into the second. Hence our general problem can be phrased: *Given a figure (or several figures), to transform same into another figure of some specified sort.*

Our two preceding papers treated the following particular instances of this general problem: To transform three congruent squares into a single square (*Problem I*), and conversely to transform a single square into three congruent squares (*Problem II*); to transform two squares into a single square (*Problem III*); to transform a square into an equilateral triangle (*Problem IV*), and conversely to transform an equilateral triangle into a square (*Problem V*); and finally, to transform a nonrectangular parallelogram into a square (*Problem VI*).

Each of these six problems was solved, i.e., appropriate cutting lines were constructed in the given figure(s), and the resulting parts were proved to rearrange into the desired figure. Problems I and II were solved in our May paper; Problems III, IV, V, and VI were solved in our October paper.

At the end of our second paper we raised, but did not answer, two questions: (a) Devise a construction by which a square is transformed into *two* congruent equilateral triangles; and (b) devise a construction by which a square is transformed into *three* congruent equilateral triangles. It is evident that these questions are two particular instances ($n=2$ and $n=3$) of:

Problem VII. Given a square, to transform it into n *congruent equilateral triangles* (where n is some natural number greater than 1 and fixed in advance).

The first part of the present paper is devoted to the solution of Problem VII. Once you have worked through this solution, you will find it easy to answer the above questions. (The case when $n=1$ is omitted from Problem VII because Problem IV has already dealt with it.)

It is clear that particular problems like the six solved so far and the seventh stated above can be multiplied indefinitely. We can, for example, ask how a

[1] These are "A Problem on the Cutting of Squares" and "More on the Cutting of Squares" in THE MATHEMATICS TEACHER, XLIX (May 1956), 332–343, and (October 1956), 442–454.

pentagon can be transformed into a square; and conversely, we can ask the same of a hexagon. But there comes a time when our experience with particular problems is extensive enough to justify confronting a general question: *Is there a systematic way by which ANY polygon can be transformed into a square?* The second part of the present paper poses this question and answers it affirmatively by formulating a sequence of steps that is guaranteed to transform any polygon into a square. An alternative procedure is possible and in this connection we show how to solve Problem VIII.

Problem VIII. Given one or more rectangles, to transform them into a square.

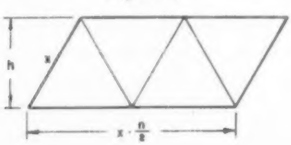
The Mathematics Student Journal for December 1956 offers another procedure that is constructive to compare the one here.

The first part (Solution of Problem VII) of the present paper can be read without knowledge of our preceding papers, though of course a knowledge of them is helpful. The second part of the present paper makes free use of all seven problems solved to that point. Here, as for our two previous papers, our main source continues to be the recent Russian book, *The Wonders of the Square*, by B. Kordiemski and N. Rusalev.

Solution of problem VII

Problem VII requires us to transform a square into n congruent equilateral triangles, where n is some natural number greater than 1 and fixed in advance. Our discussion of this problem will indicate how the square can be cut and why the resulting parts reassemble into n congruent equilateral triangles.

Figure 1



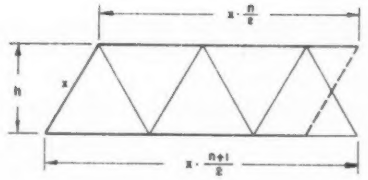


Figure 2

For a clue to the idea behind our solution, consider for a moment n congruent equilateral triangles, each having a side of length x and an altitude of length h. If the number n is even, the n triangles can evidently be arranged into a parallelogram whose altitude has length h and whose base (i.e., longer side) has length $x \cdot (n/2)$. These facts are illustrated in Figure 1 for the case $n=4$.

If, on the other hand, the number n is odd, the n triangles can be arranged into a trapezoid whose altitude has length h and whose base (longest side) has length $x \cdot (n+1)/2$. These facts are illustrated in Figure 2 for the case $n=5$. Further, it is evident from Figure 2 that if an end triangle of this trapezoid is cut along the appropriate line of midpoints (e.g., the dotted line in Figure 2), the trapezoid can be transformed into a parallelogram with an altitude of length h and a base again of length $x \cdot (n/2)$.

These remarks make visible the course of our solution of Problem VII. Having fixed the number of equilateral triangles sought, we first transform the given square into a parallelogram of suitable dimensions; thereafter, we cut this parallelogram into the desired number of equilateral triangles in the manner of Figure 1 or Figure 2.

Suppose our given square $ABCD$ has a side of length a. Suppose we wish to transform $ABCD$ into n congruent equilateral triangles, where $n \geqq 2$. Let x be the length of a side of one of these sought-for triangles and h the length of its altitude. Then, of course, $h = x \cdot (\sqrt{3}/2)$.

Our first task is to transform square $ABCD$ into a parallelogram whose alti-

tude has a length h such that $h = x\cdot(\sqrt{3}/2)$, and whose base has a length b such that $b = x\cdot(n/2)$.

Being equivalent to square $ABCD$, this desired parallelogram must have an area $b\cdot h$ such that $b\cdot h = a^2$. Since further $b = x\cdot(n/2)$ and $h = x\cdot(\sqrt{3}/2)$, it follows that

$$\left(x\cdot\frac{n}{2}\right)\cdot\left(x\cdot\frac{\sqrt{3}}{2}\right) = a^2,$$

whence

$$x^2 = \frac{4a^2}{n\sqrt{3}}$$

and so

$$x = \frac{2a}{\sqrt{n\sqrt{3}}}\,. \tag{1}$$

Equation (1) gives x in terms of the two known numbers, a and n. Consequently, both b and h can be expressed in terms of a and n:

$$b = \frac{2a}{\sqrt{n\sqrt{3}}}\cdot\frac{n}{2} = a\sqrt{\frac{n}{\sqrt{3}}}\,, \tag{2}$$

$$h = \frac{a}{\sqrt{\dfrac{n}{\sqrt{3}}}}\,. \tag{3}$$

We turn now to the matter of devising geometric constructions for segments of length b and length h respectively. Since

$$a\sqrt{\frac{n}{\sqrt{3}}} = \sqrt{\frac{a^2n}{\sqrt{3}}} = \sqrt{an\sqrt{\frac{a^2}{3}}} = \sqrt{an\sqrt{a\cdot\frac{a}{3}}}\,,$$

we see from (2) that

$$b = \sqrt{an\sqrt{a\cdot\frac{a}{3}}}\,. \tag{4}$$

Similarly, since

$$\frac{a}{\sqrt{\dfrac{n}{\sqrt{3}}}} = \sqrt{\frac{a^2\sqrt{3}}{n}} = \sqrt{\frac{a}{n}\cdot\sqrt{3a^2}} = \sqrt{\frac{a}{n}\sqrt{3a\cdot a}}\,,$$

it follows from (3) that

$$h = \sqrt{\frac{a}{n}\sqrt{3a\cdot a}}\,. \tag{5}$$

Formula (4) makes evident how a segment of length b can be obtained by two successive mean-proportional constructions: first the mean proportional between segments of lengths a and $a/3$ respectively, then the mean proportional between the segment resulting therefrom and a segment of length an. Figure 3 illustrates these steps. Having marked on AB extended the point H such that $(BH) = a/3$, we locate point R so that (BR) is the mean proportional between a and $a/3$. (Recall our convention that '(XY)' stands for the length of segment XY.) The second step of the construction calls for the mean proportional between (BR) and an. Figure 3 illustrates this step for two cases, $n=4$ and $n=5$. On AB extended mark Q such that $(AQ) = 4a$ and Q' such that $(AQ') = 5a$. Mark R' so that $(AR') = (BR)$. Finally, locate S so that (AS) is the mean proportional between (AR') and (AQ), and S' so that (AS') is the mean proportional between (AR') and (AQ'). Segment AS has the required length b for the case $n=4$, and segment AS' has the required length b for the case $n=5$.

These remarks based on formula (4) can be paralleled for formula (5). Thus we may count ourselves in possession of constructions that yield segments of lengths b and h respectively.

It is now a simple step to construct a parallelogram with dimensions b and h, and a base angle of 60°, and to cut it into n equilateral triangles according to the scheme of Figure 1 or Figure 2. A simple step—but a useless one. Our aim is to devise a cutting of square $ABCD$ into parts which reassemble into n equilateral triangles. The procedure just mentioned gives us the required n triangles, but

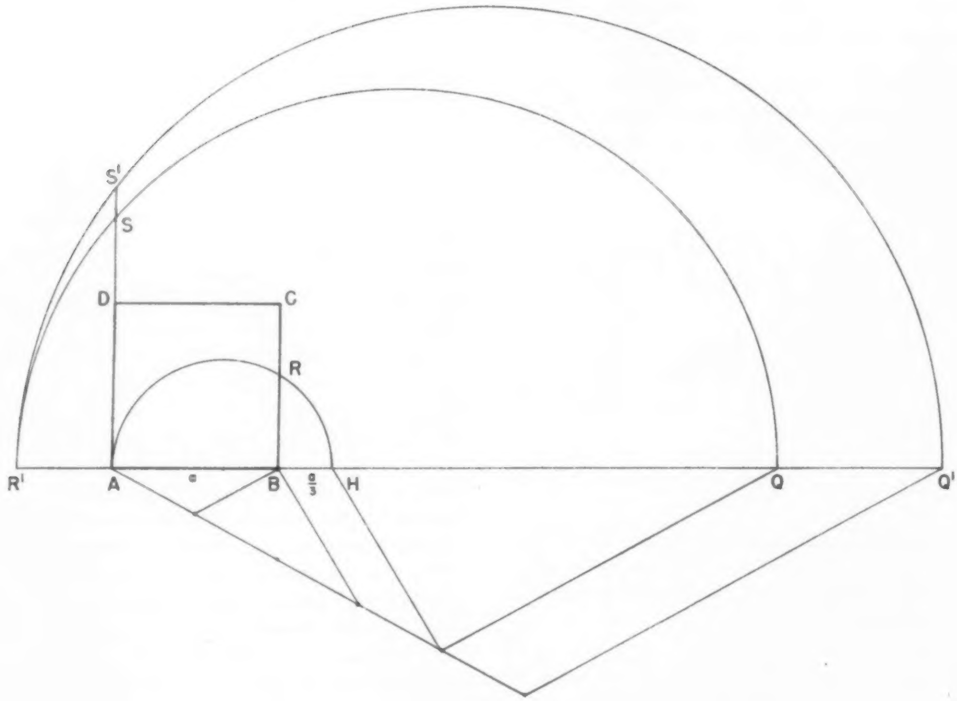


Figure 3

sheds no light whatever on how to cut square $ABCD$ into parts that reassemble into these n triangles.

Let us go back to square $ABCD$, and devise a cutting under which it transforms into a parallelogram with dimensions b and h.

Given a square $ABCD$ of side a, and given a fixed natural number n greater than 1, we have seen how to construct a segment of length b, where $b=a\sqrt{n/\sqrt{3}}$. With this segment as radius and B as center, mark on CD or its extension a point E such that $(BE)=b$, and draw segment BE. (This construction is possible i.e., provided $b>a$, i.e., provided $\sqrt{(n/\sqrt{3})}>1$, provided $n>\sqrt{3}$, i.e., provided the natural number n is 2 or greater than 2; precisely our assumption about n.) On DC mark the point F such that $(EF)=a$. Finally, draw the parallelogram $BEFG$ having segments BE and EF as two of its sides. Diagram (*i*) of Figure 4 illustrates these steps when $n=4$; Diagram (*ii*), when $n=5$.

It is evident from Figure 4 that parallelogram $BGFE$ is equivalent to square $ABCD$, for our construction of $BGFE$ supplies cutting lines in $ABCD$ which obviously divide $ABCD$ into parts that reassemble into $BGFE$. Therefore, since parallelogram $BGFE$ has a base of the desired length b prescribed in (2), it necessarily has an altitude of the desired length h prescribed *in* (3) because $b\cdot h=a^2$. Thus we have obtained a parallelogram equivalent to our original square and having the desired linear dimensions.

We remark parenthetically that our conclusions about parallelogram $BGFE$ could have been reached more quickly (but perhaps less perspicuously) as follows: De-

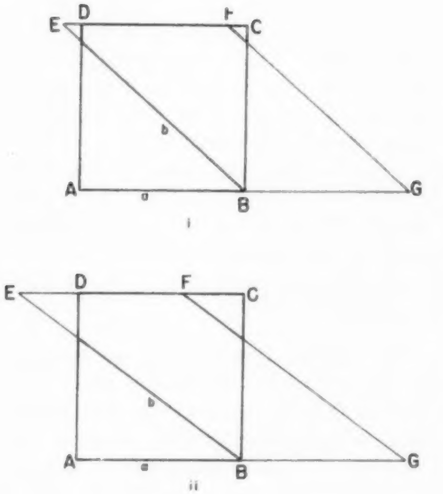


Figure 4

rive (2) and (4) as above while omitting (3) and (5). Utilize the construction of Figure 3 to obtain Figure 4; then conclude by the very last part of our solution of Problem VI[2] that parallelogram $BGFE$ is equivalent to square $ABCD$ and hence has the desired linear dimensions b and h.

While parallelogram $BGFE$ of Figure 4 has the desired linear dimensions b and h, it need not have the desired angular dimensions, viz. $\angle FEB$ may not be 60°. Our next step insures that $BGFE$ shall also have the desired angular dimensions.

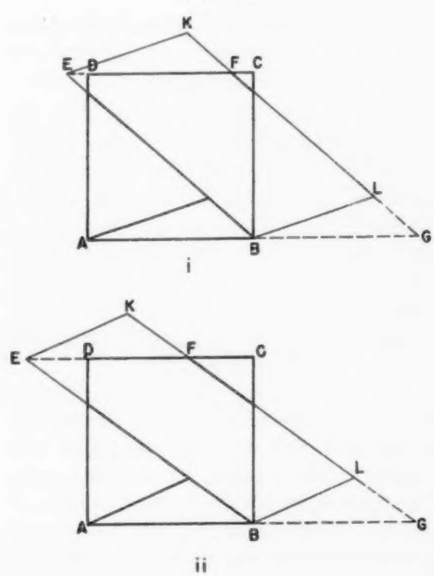
From Figure 4 we obtain Figure 5 by drawing segments EK and BL such that $\angle KEB = \angle BLF = 60°$. Diagram (*i*) of Figure 5 illustrates these steps when $n=4$; diagram (*ii*), when $n=5$. Consulting Figure 5 we see that when triangle BLG is translated to position EKF, there is obtained a parallelogram $BLKE$ with the desired linear and angular dimensions. The segment from A to BE parallel to BL indicates the extra cut required to divide square $ABCD$ into parts that reassemble into parallelogram $BLKE$.

[2] See "More on the Cutting of Squares," The Mathematics Teacher, XLIX (October 1956, 453).

The solution of our original problem is now quickly completed. In the parallelogram $BLKE$ draw segments which divide $BLKE$ into n congruent equilateral triangles. If n is even, the division comes out exactly (recall Figure 1); if n is odd, we first obtain from $BLKE$ $n-1$ equilateral triangles and then get the desired extra triangle by dividing the small residual parallelogram appropriately (recall Figure 2). Diagram (i) of Figure 6 illustrates this step when $n=4$; diagram (ii), when $n=5$. Notice in diagram (ii) that the desired fifth triangle is obtained by drawing in parallelogram $BLMQ$ the segment MN from M to the midpoint N of BL and rotating triangle NLM into postion BPN.

Our final step is to add to square $ABCD$ the extra segments which complete the partition of $ABCD$ into parts that reassemble into the desired n equilateral triangles. This addition is a simple matter. We illustrate it in Figure 7, labelling congruent parts with the same numeral. Diagram (i) of Figure 7 thus shows us how square $ABCD$ may be cut into *nine* parts which reassemble into *four* congruent

Figure 5



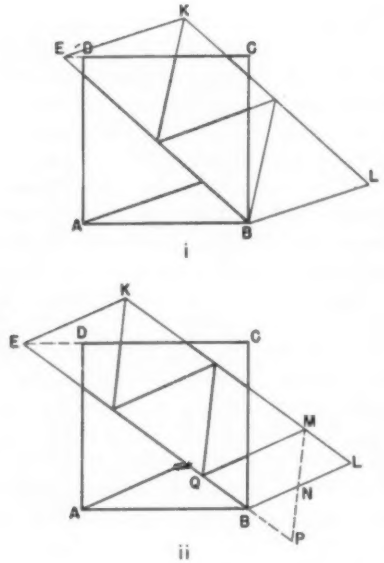


Figure 6

equilateral triangles. Diagram (ii) similarly shows how $ABCD$ may be cut into *eleven* parts which reassemble into *five* such triangles. Our solution of Problem VII is now complete.

Preliminaries to the general question of transforming a polygon into a square

Basic to our whole discussion of the cutting of figures to obtain other figures is the fundamental theorem of geometry cited at the outset of our first paper: *If two given polygons have equal areas, then either of them can be subdivided into a finite number of polygonal parts which, upon rearrangement, form the other given polygon* (i.e., form a polygon congruent to the other given polygon).

This theorem tells us that *in principle* it is always possible to solve the general problem stated at the beginning of this paper: *Given a figure, to transform same into another figure of some specified type.* A solution of this general problem comprises a *construction* of appropriate cutting lines in the given figure, and a *proof* that the resulting parts reassemble into the desired sort of figure. Now it is one thing to know there must be a solution, but another to produce a solution, and still another to establish a "best" solution, the one requiring the smallest number of cuts in the given figure. The remainder of the present paper will center around a *constructive* proof of the theorem:

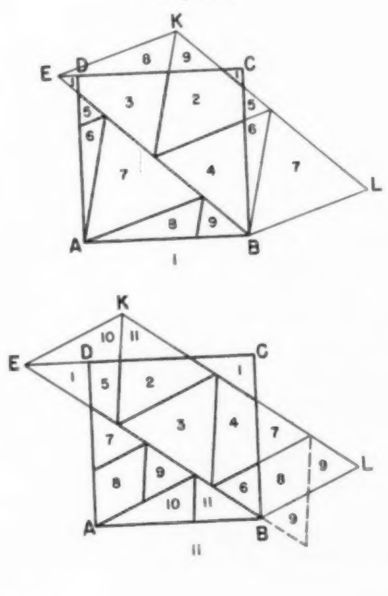
Any polygon can be transformed into a square.

By proving this theorem we establish an important special case of the fundamental theorem cited above. By proving this theorem *constructively* we show *how* by an explicit method *to produce* a solution of an important special case of the general problem stated above.

First, let us be clear about the meaning of the term "polygon" in our theorem. All of us know that *a polygon is a closed figure bounded by straight segments and comprising all the points in its interior and on its boundary.* Some examples of polygons are shown in Figure 8.

Now, of the polygons shown in Figure

Figure 7



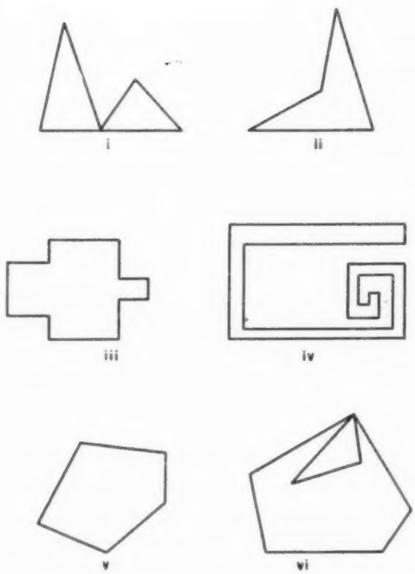


Figure 8

8, some have boundaries that divide the plane into exactly two regions and some have boundaries that divide the plane into more than two regions. Thus each of polygons (ii), (iii), (iv), and (v) has a boundary that divides the plane into exactly two regions—an interior and an exterior. The boundary of polygon (i) divides the plane into three regions—two interiors and one exterior. The boundary of polygon (vi) also divides the plane into three regions—one interior and two exterior. (To decide into how many regions the boundary of a polygon divides the plane, simply imagine the plane to be cut along the segments that compose the boundary of the polygon. The number of separate pieces that result is the number of regions into which the boundary divides the plane.)

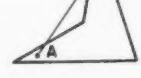
When the term "polygon" is used in elementary geometry, what is meant is a polygon whose boundary divides the plane into *exactly two* regions. Let us call such polygons *ordinary polygons.* Thus polygons (ii) (iii), (iv), and (v) of Figure 8 are ordinary polygons, while polygons (i) and (vi) are not. The theorem we are now discussing is valid when "polygon" is understood to mean "ordinary polygon." However, the proof we shall give below of this theorem only applies to a special kind of ordinary polygon. What this special kind is we now make clear.

In a polygon pick any two points, and draw the segment joining them. This segment may lie entirely within the polygon under consideration, or it may not. If for *every* pair of points in a polygon, the segment joining them lies within that polygon, the polygon is said to be *convex.* Of the polygons in Figure 8, only polygon (v) is convex. Polygon (ii) of Figure 8 is *not* convex because we can find in it two points (e.g., A and B in Figure 9) whose connecting segment passes outside the polygon. There are numerous examples of convex polygons in addition to (v) in Figure 8. E.g., any triangle is convex, and so likewise is any parallelogram.

There is a beautiful mathematical theory of convex figures which we hope we may someday present in the pages of THE MATHEMATICS TEACHER. We need essentially nothing of that theory now, however, and content ourselves with the obvious remark that *if a polygon is convex, then it is ordinary.* (The converse of this remark is false, as Figure 9 makes plain.)

The theorem we shall now prove runs: *Any convex polygon can be transformed into a square.* Which is to say, in the statement of this theorem displayed four paragraphs back, we understand "polygon" to mean "convex polygon." While the convex polygons are a special kind of ordinary polygon, all of the familiar polygons (triangles, parallelograms, squares, etc.) are convex—hence a proof of this theorem is of considerable value.

Figure 9



Before launching into the proof, let us give a synopsis of it. Basically, our proof consists in working out one of the possible schemes for transforming an *arbitrary* convex polygon into a square. The important stages in the chosen scheme of transformation are singled out as *lemmas*, or auxiliary theorems. First we show that (1) *any convex polygon can be partitioned into some definite number of triangles* and that (2) *any triangle can be divided into parts which reassemble into a parallelogram.* In view of (2), each of the triangles obtained from the original convex polygon in accordance with (1) can be transformed into a parallelogram. We have already learned how (3) *each parallelogram can be transformed into a square.* Hence, once we show that (4) *if each of two polygons separately can be transformed into the same third polygon, then the first polygon can be transformed into the second,* it follows from (2) and (3) that (5) *any triangle can be transformed into a square.* Thus each of the triangles obtained from our original convex polygon can be transformed into a square. These squares are then composed into a single square with the help of the fact that (6) *any two squares can be transformed into a single square.*

A PROOF OF THE THEOREM THAT ANY CONVEX POLYGON CAN BE TRANSFORMED INTO A SQUARE

We now prove the theorem: *Any convex polygon can be transformed into a square,* by following out the scheme of transformation just outlined.

Lemma 1. *Any convex polygon can be partitioned into some definite number of triangles.*

Proof. Pick a vertex of any convex polygon and draw all the diagonals of the polygon that lead from that vertex. (Any such *diagonal* is a segment which is not a side and which leads from the selected vertex to some other vertex of the polygon). These diagonals divide the convex polygon into triangles. If the original convex polygon has n sides, then it has n vertices, $n-3$ diagonals from a given vertex, and falls into $n-2$ triangles when cut along these diagonals. The number $n-2$ is the "some definite number" referred to in the lemma. These considerations are illustrated in Figure 10 for the case $n=6$.

Lemma 2. *Any triangle can be transformed into a nonrectangular parallelogram whose base is congruent to some side of the triangle and whose altitude has a length half that of the corresponding altitude of the triangle.*

Proof. In an arbitrary triangle ABC (see Figure 11) mark midpoint D of AC and midpoint E of BC. Draw segment DE and extend same to its intersection F with the line BF_1 drawn through vertex B parallel to AC. Triangles CDE and BFE clearly are congruent. Hence the parallelogram $ABFD$ is equivalent to (has the same area as) triangle ABC. Further, parallelogram $ABFD$ has for its base the side AB of the given triangle and an altitude whose length is half the length of the altitude of triangle ABC to AB.

If triangle ABC is not a right triangle, parallelogram $ABFD$ is obviously nonrectangular. If triangle ABC is a right triangle, parallelogram $ABFD$ produced by our construction may be rectangular—

Figure 10

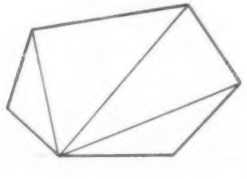
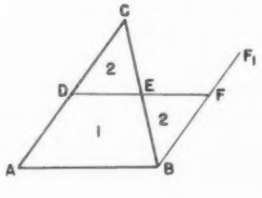


Figure 11



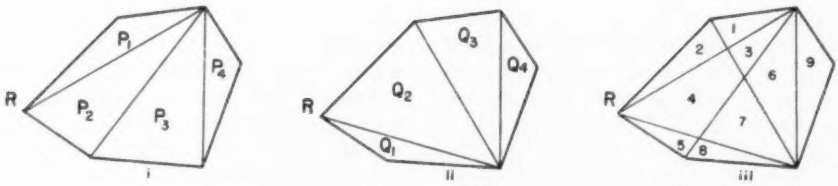


Figure 12

but will not be if the vertex having the right angle is labelled "C" (which can always be arranged by relabelling the vertices).

Lemma 3. *Any nonrectangular parallelogram can be transformed into a square.*

Proof. The proof of this lemma consists in the solution of Problem VI given in our previous paper, "More on the Cutting of Squares."

Lemma 4. *If each of two polygons separately can be transformed into the same third polygon, then the first polygon can be transformed into the second.*

Proof. Suppose we are given two polygons, P and Q. Suppose P can be transformed into a third polygon R, and that Q can also be transformed into R. Which is to say, suppose P and R can each be cut into parts that are pairwise congruent, and likewise Q and R can each be cut into pairwise congruent parts. (The component parts of P and of R may, of course, bear no resemblance to the component parts of Q and R; also the number of parts of P and R may differ from the number of parts of Q and R.)

Now refer to Figure 12, which shows three copies of polygon R. Diagram (i) of Figure 12 represents a possible cutting of R into parts that reassemble into the polygon P not shown in the figure. These parts are labelled P_1, P_2, P_3, and P_4 respectively. Diagram (ii) represents a possible cutting of R into parts that reassemble into polygon Q not shown in the figure. These parts are labelled Q_1, Q_2, Q_3, and Q_4 respectively.

In polygon R let both systems of cutting lines, that of diagram (i) and that of diagram (ii), be drawn and the result is that shown in diagram (iii) of Figure 12. From the larger number of smaller parts obtained in diagram (iii), one arrangement can produce each of parts P_1, P_2, P_3, and P_4 of polygon P (e.g., parts 1 and 2 together compose P_1, parts 3 and 4 and 5 together compose P_2, etc.) and hence the complete polygon P. Another arrangement can produce each of parts Q_1, Q_2, Q_3, and Q_4 of polygon Q (parts 5 and 8 together compose Q_1, parts 2, 4, and 7 compose Q_2, etc.) and hence the complete polygon Q.

Thus, using the same parts of R but arranging them differently, we can obtain from R both the polygon P and the polygon Q. It follows from this that polygon P can be transformed into polygon Q, i.e., that P can be suitably partitioned and the resulting parts rearranged into Q.

Lemma 5. *Any triangle can be transformed into a square.*

Proof. This result follows from Lemmas 2, 3 and 4.

Lemma 6. *Any two squares can be transformed into a single square.*

Proof. The proof of this lemma consists in the solution of Problem III given in "More on the Cutting of Squares.

Together, these six lemmas establish that any convex polygon can be transformed into a square. Our proof is complete.

Miscellaneous comments on the theorem and its proof

The six lemmas of our proof can be regarded as six steps in a *method* by which we pass from a convex polygon to an

equivalent square. Our proof thus constitutes an *algorism* for solving the general problem of transforming a convex polygon into a square. It is because of this algorismic feature that we say our proof is a *constructive* one.

Now clearly if one figure can be transformed into a second, then by that very fact the second figure can be transformed into the first. Thus from our theorem it follows that *a square can be transformed into a convex polygon* of any specified shape. Note that the proof of this conclusion is *not* a constructive one.

Suppose we are given a convex polygon and asked to transform it into a square. We could follow the six steps of our general method (the six lemmas constituting our proof) and at the end produce the desired square. Almost always, however, this construction entails many superfluous cuts. In each concrete case, therefore, there remains the problem of constructing the square with fewer cuts.

Although we have *proved* that any convex polygon can be transformed into a square, we have observed that any *ordinary* polygon can be transformed into a square. Our proof fails to establish the more general result about ordinary polygons precisely because our *proof* of Lemma 1 is valid only for convex polygons. It is possible to prove constructively a stronger version of Lemma 1: *Any ordinary polygon can be partitioned into some definite number of triangles.* Once this is done, our remaining lemmas 2 through 6 combine with it to give a proof of the more general theorem. We have not proved this stronger version of Lemma 1 because the proof is based on considerations lying outside the familiar tradition of plane geometry.

A different proof of our theorem

Another systematic way to transform an arbitrary convex polygon into a square follows. (1) Cut the convex polygon into triangles. (2) Transform (by one cut) each triangle into a parallelogram. (3) Transform (by one cut) each parallelogram into a rectangle. (4) Transform all these rectangles into rectangles having congruent bases, and finally (5) stack these rectangles into one large rectangle and transform this last into a square.

This scheme of transformation provides a basis for a different constructive proof of our theorem that any convex polygon can be transformed into a square. As earlier, so here important stages in the proof can be summarized into lemmas.

The first stage sees the use of lemma 1, already in hand. The second stage requires lemma 2, also already established. Actually, in view of the third stage, we can make free use of the method expressed in the proof of lemma 2 without concern as to whether it produces a rectangular parallelogram. The third stage requires a new lemma to the effect that any parallelogram can be transformed into a rectangle; however, the construction and proof behind this transformation are so simple (see Figure 13) that we shall not dignify it into a lemma.

Figure 13

It is in the fourth and fifth stages that we reach nontrivial questions requiring attention. We treat these questions by stating their essential element as lemmas and then proving them. Once these matters are in our possession, so is the new proof of our theorem.

Stage four requires that we transform several rectangles into rectangles having congruent bases. The key here is the following lemma:

Lemma 4a. *Any rectangle can be transformed into another rectangle, one of whose sides has an arbitrary prescribed length.*

Proof. We divide our proof into five cases.

Case (i). Suppose we are given rectangle $ABCD$, and suppose the equivalent rectangle sought is to have a side whose prescribed length is *greater* than that of

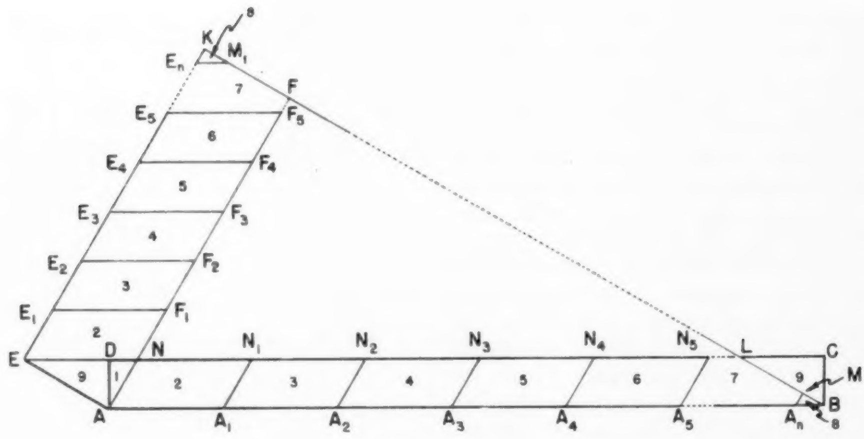


Figure 14

the smaller side of $ABCD$ but *less* than that of the diagonal of $ABCD$.

We now construct Figure 14 in the following way. On the extension of side CD mark point E such that segment AE is congruent to the prescribed side of the desired rectangle. From vertex B draw a line parallel to AE. From A and from E draw segments AF and EK perpendicular to the parallel just drawn through B. Let the points of intersection of CD with BK and AF be labelled L and N. Next, observing there is a unique natural number n such that[3]

$$n\cdot(EN)\leqq(AB)<(n+1)\cdot EN),$$

let us begin at vertex A and on side AB lay off the segment EN successively n times. Let $A_1, A_2, A_3, \cdots, A_n$ be the successive points of subdivision and from them draw segments parallel to AF. The parallels from $A_1, A_2, \cdots, A_{n-1}$ cut LD at points $N_1, N_2, \cdots, N_{n-1}$ respectively. The parallel from A_n cuts BL at the point M. Should it be the case that $n\cdot(EN)=(AB)$, then both A_n and M coincide with vertex B. Finally, begin at E and in the same way lay off on EK the segment AN successively (it can be laid off n times, too). Take $E_1, E_2, E_3, \cdots, E_n$ to be the successive points of subdivision, and from them draw segments parallel to EN. The parallels from $E_1, E_2, \cdots, E_{n-1}$ meet AF at $F_1, F_2, \cdots, F_{n-1}$ respectively. The parallel from E_n meets KF at M_1. Again, E_n and M_1 will both coincide with K should it be the case that $n\cdot(EN)=(AB)$.

It follows from the construction that the parts of Figure 14 labelled with the same numerals are congruent. The part labelled 1 is common to both rectangles $ABCD$ and $AFKE$. Hence rectangles $ABCD$ and $AFKE$ are equivalent, where $AFKE$ has a side (it is EA) of the prescribed length.

Case (ii). Now suppose the equivalent rectangle sought is to have a side whose prescribed length is *greater* than that of the diagonal of the given square.

To use the same Figure 14, let us suppose here that $AFKE$ is the given rectangle. Extend KF to the point B such that AB has the length prescribed for a side of the rectangle sought. Draw EC parallel to AB, and both AD and BC perpendicular to EC. On sides EK and AF lay off from E and from A successive seg-

[3] The existence and uniqueness of this number n go back to the foundations of the theory of segment measurement. The critical item there is the so-called *Axiom of Archimedes*, which says: If segments XY and UV are such that $(XY)<(UV)$, then there is a natural number n such that $n\cdot(XY)\leqq(UV)<(n+1)\cdot(XY)$. [Recall again that (XY) is the length of segment XY.]

ments equal to AN, and draw segments E_1F_1, E_2F_2, etc. With the help of analogous constructions, matching (congruent) parts can be drawn in $ABCD$ and our construction of the cutting lines in $AFKE$ verified.

Case (iii). Finally, suppose the equivalent rectangle sought is to have a side whose prescribed length is *smaller* than that of either side of the given rectangle.

Again, take $AFKE$ of Figure 14 to be the given rectangle. Through vertex E draw a line EC whose perpendicular distance from A is the length prescribed. (This means that in Figure 14 we are assuming segment AD to have the prescribed length.) Constructions from here on are analogous to previous ones.

Cases (iv) and (v). The two remaining cases are: (iv) the prescribed length is the *same* as that of the shorter side of the given rectangle; and (v) the prescribed length is the *same* as that of the diagonal of the given rectangle. Case (iv) clearly presents no problem since it requires no transformation. As for case (v), if the prescribed side is congruent to the diagonal of the given square, then the other side of the rectangle sought must have a length less than that of either side of the given rectangle. Thus we need only construct a segment whose length is that of the shorter side of the desired rectangle, and then use this shorter side in the fashion of case (iii). Figure 15 shows a construction for the shorter side. Presupposing that the given rectangle is $ABCD$ and that the diagonal AC thereof is congruent to the prescribed side, the construction in Figure 15 furnishes the desired shorter segment AX such that $(AX)\cdot(AC)=(AD)\cdot(AB)$.

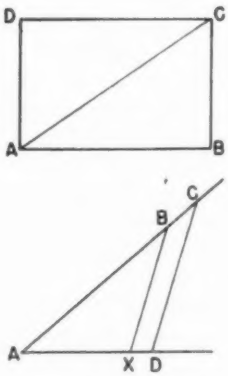


Figure 15

Lemma 5a. *Any rectangle can be transformed into a square.*

Proof. Given a rectangle $ABCD$ (see Figure 15), its area is $(AD)\cdot(AB)$ and so the side of the square equivalent to $ABCD$ must have a length $\sqrt{(AD)\cdot(AB)}$. We construct a segment of this length using the familiar mean-proportional construction. Since the length $\sqrt{(AD)\cdot(AB)}$ is clearly greater than that of the shorter side of $ABCD$ and at the same time shorter than that of the diagonal of $ABCD$, we are in the situation treated under case (i) of the proof of Lemma 4a. Applying the method of case (i), we transformed $ABCD$ into a square as desired.

Lemmas 4a and 5a provide the necessary keys to stages four and five of our alternative scheme of transformation. They thus enable us to give an alternative proof of the theorem that any convex polygon can be transformed into a square.

We remark, finally, that the proof of Lemma 5a amounts to a solution of the problem of transforming a rectangle into a square; and proofs of Lemmas 4a and 5a together show how to transform several rectangles into a square. Thus we also have a solution to Problem VIII.

Problem VIII. Given one or more rectangles, to transform them into a square.

Concluding comment

The three papers of our series have presented solutions of eight problems involving the transformations of figures. The methods of these solutions, together with the theorem and its proofs, give us an approach to any problem of this general type. Our task now is to confirm our knowledge and skill by exercises. We shall turn to such exercises in our next paper.

What do we mean?[1]

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An examination of the meaning of (1) some key mathematical words frequently used in mathematics classes, and (2) some words not so frequently used but which are rapidly becoming known and used in secondary circles.

EVERY TEACHER knows that it is easier to teach skills than to teach concepts. Moreover, skills are easier to evaluate, and their acquisition gives a student, his teacher, and his parents a glow of achievement. But sometimes the amount of real achievement is illusory. Mathematics has often been called a system of ideas. To slight the ideas is to deprive the subject of its essence, regardless of how many correct answers may have been obtained.

All of us have at times entered upon the boggy ground of the "good enough." We settle for a definition that is in fact no definition, or we accept a form of solution that is incorrect but easy to write down, or we drill at the steps of a process without giving adequate insight into what is going on. Of course we know that this is inexcusable, but time is short, we are busy, and the students are slow. Then comes the day of questioning: Student A can find no mistake in his solving a certain equation, but the check fails to work. What has happened to the algebra? Student B wonders why he must prove two theorems in his locus problems. Doesn't the second theorem always turn out to be true? Student C can draw a beautiful graph, but he can get no information from it. Why do the co-ordinates of a point of intersection satisfy both equations of the system?

The authors venture the opinion that sometimes fewer problems done with more attention to the concepts behind the processes would produce more worth-while results. The purpose of the present article is to attempt to show how the idea of *necessary and sufficient* or *if and only if* conditions may add more meaning and insight to common topics in secondary school mathematics.

NECESSARY AND SUFFICIENT CONDITIONS

One of the difficult ideas for students of mathematics to appreciate and to use effectively is that of necessary and sufficient conditions. What is a *necessary condition?* What is a *sufficient condition?* In fact, just what is a *condition?* Conditions play an important role in mathematics. It seems as if it would be well for us to investigate answers to each of these questions.

Some mathematicians say that the language of "necessary and sufficient" is losing ground and that the language of "if and only if" is being substituted. This is probably true, and yet there is still wide use of the "necessary and sufficient" language; there seem to be situations where it is more natural than the "if and only if"

[1] The writers wish to express their appreciation of the suggestions made by all those who read an earlier draft of this paper. In addition, they are indebted to Professor Robert M. Exner, Syracuse University, for a number of pertinent comments.

language. We shall attempt in our discussion to use both means of describing certain statements. Then a reader can make his own choice between them.

As we attempt to analyze the use of the word "condition" in mathematics, it seems that a condition is expressed by a sentence in which there are occurrences of some sort of *place holder*. For example, the check form of a bank checking account contains a condition, "Pay to the order of ________________ $______," in which the blanks are place holders. Another sort of place holder is x in the condition $x<4$. (In mathematical logic such place holders are called occurrences of *free variables.*) A condition is not a statement, a sentence that is either true or false. Thus, the sentence,

r is a rectangle,

is a perfectly good sentence; it is also a condition. But we cannot tell whether it is true or false until we have more information about the place holder r. However, if a particular name is substituted for r, then the condition becomes an ordinary statement, which is either true or false. That is, if for the place holder r in the foregoing sentence we substitute the name $ABCD$ of a square, then we have a statement,

$ABCD$ is a rectangle,

which we know to be true from certain facts of Euclidean plane geometry.

As another example, consider the equation

$$x^2-3x+4=0.$$

The equation is a sentence that is neither true nor false; we say it is a condition on x. When we substitute a number for x, we get a statement and we can say that the resulting statement is either true or false.

With the foregoing discussion and description of a condition, we are ready to investigate "necessary condition" and "sufficient condition" usage in mathematics. In what follows, we shall understand by q a variable whose range is all quadrilaterals. Furthermore, all substitutions for q will be names of quadrilaterals. Such a substitution will be called a *substitution instance.*

Suppose we have the following two conditions on q:

(1) q is a square, and

(2) each of q's angles is a right angle.

Then, to say that (2) is a necessary condition for (1) is to assert:

For all q, if q is a square, then each of q's angles is a right angle.

Or, we can write:

For all q, a necessary condition for q to be a square is that each of q's angles be a right angle.

However, (2) is not a sufficient condition; it does not guarantee that "q is a square" will be satisfied for all substitution instances of q. When we make this assertion, we are saying:

For some q, each of q's angles is a right angle *and* q is not a square.

Clearly, if $ABCD$ is the name of a rectangle and is substituted for q the assertion is true.

In order to exhibit a sufficient condition, let us introduce another condition:

(3) q has four equal sides.

Now, we can assert:

For all q, (2) and (3) imply (1).[2]

[2] We follow the practice, which seems quite prevalent in mathematics, of using various forms of a sentence interchangeably. All of these forms are considered to have the same meaning. For example, the sentence, "If n is an even number, then n is exactly divisible by 2," might appear in any of the forms: (a) n is an even number *implies* n is exactly divisible by 2; (b) n is an even number *only if* n is exactly divisible by 2; (c) n is exactly divisible by 2 is a *necessary condition* for n to be an even number; (d) n is an even number is a *sufficient condition* for n to be exactly divisible by 2; (e) n is exactly divisible by 2 *if* n is an even number.

Or, stated in an equivalent form:

> For all q, if each of q's angles is a right angle and q has four equal sides, then q is a square.

Thus, (2) and (3) are sufficient for (1). It is clear that (2) and (3) are also necessary, since

(1) implies (2) and (3).

That is, (2) and (3) are necessary and sufficient to guarantee that "q is a square" will be satisfied for all substitutions for q. In other words:

> For all q, q is a square if and only if each of q's angles is a right angle and q has four equal sides.

Two examples will serve to indicate the strength of conditions. If a quadrilateral has four 10-inch sides and each of its angles is a right angle, these conditions are sufficient to make the quadrilateral a square. But these conditions are not necessary, for the sides of the quadrilateral might each be 5 feet long and the figure would still be a square.

Suppose:

> A man can be admitted to a show if and only if he has 90¢.

A man's possession of 75¢ is a necessary condition for his admission, but it is not sufficient. On the other hand, a man's possession of \$5 is a sufficient condition for his admission but not necessary. A man's possession of 90¢ is both necessary and sufficient for his admission to the show.

Locus

Pedagogically a concept of locus easily appreciated by high school students is that of a moving point—the so-called dynamic approach to locus. However, logically the static approach is the sounder of the two. According to the latter concept, a locus is a set of points satisfying a certain condition. The locus consists of all those points and only those points satisfying the condition. This description of a locus asserts:

> If a point is on the locus, then the point satisfies the condition of the locus,

and

> if a point satisfies the condition of the locus, then it is on the locus.

In other words, the description of a locus problem specifies a condition that it is necessary and sufficient for all points belonging to the locus to satisfy. To make the discussion clearer, let us consider a specific statement:

> What is the locus of a point equidistant from two intersecting lines?

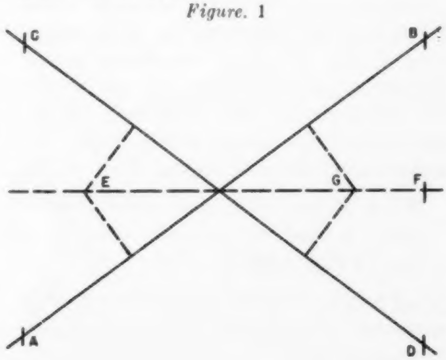
Everyone knows the correct answer, but let us examine the statement from the point of view of a careless plane-geometry student. Such a student might be satisfied with Figure 1. He might say to himself, "Points E and G are equidistant from AB and CD. These points seem to lie on the bisector of the angle between lines AB and CD. Therefore, I must prove that the bisector EG is the locus." Now, if he tries to prove this conjecture, how far can he get? He might show, for any point F:

> If F is on the bisector EG, then F is equidistant from AB and CD.

But how does he know that he may not have missed some points? He does not know, since he has not shown that, for any point F:

> If F is equidistant from AB and CD, then F is on EG.

Figure. 1



A student with more insight might point out that there are two pairs of angles between AB and CD, and he might add the bisector of the second pair of angles to his description of the locus. This student is in a position to show that the two bisectors make up the correct locus. There are no extra points and there are no points missing. The second student can now prove, for any point F:

> If F is on a bisector, then F is equidistant from AB and CD,

and

> if F is equidistant from AB and CD, then F is on a bisector.

Why does a locus statement assert two statements to prove? It seems to us that the answer lies in having to reply to the question: How do you know that you have not missed some points satisfying the condition of the locus or included some points that do not satisfy the condition of the locus? The question can be answered only by establishing the sufficiency of the stated condition of the locus in addition to its necessity.

The foregoing example illustrates one point that we wished to make. If the assertion of the sufficient condition is not proved, some points satisfying the condition of the locus may be missed. That is, for a locus we must always prove an assertion of the following sort. For any point F: if F satisfies the condition of the locus, then F lies on the locus.

Consider the following example which illustrates our second point. If the assertion of the necessary condition is not proved, some points that do not satisfy the condition of the locus may be included in a description of the locus. The statement of the problem is:

> What is the locus of a point P at which a fixed line segment AB subtends a right angle?

Most people would say the locus is a circle with AB as diameter. It is true that, for any angle APB:

> If angle APB is 90°, then P is on the circle with AB as diameter.

That is, the sufficient condition is satisfied. For any point P: if P satisfies the condition of the locus, then P is on the locus. However, consider the necessary condition. For any angle APB:

> If P is on the circle with diameter AB, then angle APB is a right angle.

A careful examination of the statement shows that the circle described contains two points P for which the condition of the locus is not satisfied. These are the end-points of the given segment AB; when P is at either A or B, angle APB does not exist.

The statements concerning a locus problem and the statements concerning the graph of an equation use similar language. Of course, we should expect the similarity in language because the graph of an equation can be described as follows: The graph of an equation consists of those points and only those points whose co-ordinates satisfy the equation.

In analytic geometry we usually begin the derivation of the equation of a circle, for example, by saying:

> A circle is the locus of a point at a given distance from a given fixed point.

If the given point is the origin and the given distance is r units, we then show that $x^2+y^2=r^2$ is the equation of the circle. What we have done is to prove two statements. For any point P whose co-ordinates are (x, y):

> If the co-ordinates of P satisfy $x^2+y^2=r^2$, then P is at distance r units from the origin,

and

> if P is at distance r units from the origin, then the co-ordinates of P satisfy $x^2+y^2=r^2$.

A statement equivalent to the foregoing two statements follows. For any point P

with co-ordinates (x, y):

> The co-ordinates of P satisfy $x^2 + y^2 = r^2$ if and only if P is at distance r units from the origin.

We have discussed the use of "necessary and sufficient" language in geometry. On occasion, if some sort of language is to be introduced in a secondary school classroom, the "if and only if" phraseology may be preferable. However, it is probably even better to use the conjunction of the explicit statements. For example, for all quadrilaterals q:

> q is a square implies q has its angles right angles and its sides equal,

and

> q with its angles right angles and its sides equal imply q is a square.

Thus, a student becomes familiar with the concept of "if and only if" conditions and does not have to worry about other terminology until later mathematics courses.

Solution of equations in algebra

Consider the problem of finding a root of the equation $2x-3=7$. Now, $2x-3=7$ in itself is not a statement; you cannot tell whether it is true or false. But it is a condition concerning the unknown x, for which we shall agree to use only real numbers as substitution instances. There is a question of what is meant by words such as "solve," "solution," and "root." It seems to us when a direction is given "solve the equation" that a person is talking about the process, or the procedure, that is used to find a root. On the other hand "solution" seems to be used with at least two meanings in mathematics. Sometimes "solution" refers to the procedure used to solve the given equation, and sometimes "solution" is used as another name for a root. Most people—and writers—agree on a definition of "root." For convenience of notation, let us restrict ourselves to defining what is meant by a root r of the equation $P(x)=0$, where $P(x)$ denotes a polynomial. Then, r is a root of $P(x)=0$ if and only if $P(r)=0$. The statement means:

> r is a root of $P(x)=0$ implies $P(r)=0$,

and

> $P(r)=0$ implies r is a root of $P(x)=0$.

Notice that nothing at all is said about how r is obtained. It may have been guessed, gotten by the method of false position, or derived in the usual way.

Let us examine the obtaining of r "in the usual way." If a student is faced with the problem (or exercise!) of solving a linear equation like $2x-3=7$, he typically writes:

$$2x-3=7$$
$$2x=10$$
$$x=5.$$

Then he stops, unless his teacher insists that he do a check. The question is, does the student know the meaning of what he has done? By correct transformations, which we shall not go into here, the logic seems to be:

$$2x-3=7$$

implies

$$2x=10;$$

which in turn implies

$$x=5.$$

Then, using the generalization principle,[3] we can write

> For any number x: $2x-3=7$ implies $x=5$.

By correct transformations the given equation has been transformed into the

[3] Myron F. Rosskopf and Robert M. Exner, "Some Concepts of Logic and Their Application in Elementary Mathematics," The Mathematics Teacher, XLVIII (May 1955), 293-297. In this paper there is a simple explanation of the generalization principle and several illustrations of its use in algebra and plane geometry.

equation $x=5$. This equation is still a condition because we cannot tell whether it is true or false. However, a glance tells us what its only true substitution instance is; it is 5. Now we have a number that we can test to see if it is a true substitution instance of the given equation. Or, to put the matter another way, we have a number 5 that we know from the generalization arrived at to be the unique number that *may be* a root of the given equation. To see if 5 is a root, we must substitute 5 for x in the given equation, a condition, to see if the resulting statement is true.

We write:

Substitute 5 for x.
Left member is $2(5)-3=7$.
Right member is 7.

Since the left member is the same number as the right member, 5 satisfies the equation and is a root.

First, notice that the word "check" is not used at all. Through its careless use, we believe an idea has crept into the minds of some students that to check an equation is merely to see if the arithmetic is correct. Nothing could be further from the truth. The so-called check is an integral part of the solution and shows that the number we have found does actually satisfy the given equation, or does not.

As a second example, consider the equation:

$$\frac{3}{x}+\frac{2x-1}{x-1}=\frac{1}{x(x-1)}-1.$$

We solve this equation as follows:

(4) $$\frac{3}{x}+\frac{2x-1}{x-1}=\frac{1}{x(x-1)}-1$$

and $x\neq 0$ and $x\neq 1$

implies $3(x-1)+x(2x-1)$
$=1-x(x-1);$

implies $3x^2+x-4=0;$

implies $3x+4=0$ or $x-1=0;$

(5) implies $x=-4/3$ or $x=1$.

By the generalization principle;

(6) For any number x: (4) implies (5).

As in our first example, the given condition on x has been "simplified"; (5) is just a disjunction of two simpler conditions, so simple in fact that their true substitution instances are obvious. Is it possible that the first step in the procedure, multiplying each term of the given equation by $x(x-1)$, introduced a number that is a true substitution instance of subsequent equations but is not a true substitution instance of the given equation? To put the problem another way, are $-4/3$ and 1 both roots of (4), or is just one of these numbers a root, or is neither a root? Of course, 1 as a possible root has been eliminated already since $x\neq 1$ is a part of our hypothesis in (4). However, we shall proceed as if we failed to notice this fact.

There are two investigations that must be made before we can decide whether $-4/3$ or 1 are roots. We proceed thus:

Substitute $-4/3$ for x.
Left member is $-9/4+11/7=-19/28$
Right member is $9/28-1=-19/28$.

Since the left member is the same number as the right member, $-4/3$ satisfies the equation and is a root.

Substitute 1 for x.
Left member is $3+1/0$.

But division by zero is undefined; therefore, $1/0$ is meaningless. Hence, 1 cannot satisfy the equation, and 1 is not a root.

At this point some authors and teachers are not very clear in their efforts to explain to students the reason for the failure of 1 to satisfy the equation. Would it not be simpler—and clearer conceptually—to lead students to understand what it means to say that one has solved an equation and found a number that is a root of the equation?

When we have proved (6), we assert that the numbers that *may* satisfy the given equation must be $-4/3$ or 1. Now,

to find if one of these numbers actually satisfies the equation and can thus be called a root of the equation, we must ask ourselves, Does $-4/3$ or 1 satisfy the given equation?

Many simple examples must be used in order to make statements concerning the roots of an equation part of the thinking of students. For example, in quadratic equations we might take time to write: For any number x,

$$x^2-4=0 \text{ implies } x=+2 \text{ or } x=-2,$$

and

$$x=+2 \text{ or } x=-2 \text{ implies } x^2-4=0.$$

Without many such examples, a student is apt to lose sight of the concept among a mass of details.

Another place in algebra where it is crucial to realize what is involved when a number is called a root of an equation is in the solving of radical equations. Here, too, the check serves to clinch whether a number satisfies the given equation and, therefore is a root, or does not satisfy the given equation and therefore is not a root.

Consider, for example, the solution of the following radical equation:

$$\sqrt{3x-5}=1+\sqrt{2x}$$

implies $\quad x-6=2\sqrt{2x};$

implies $\quad x^2-20x+36=0;$

implies $\quad x=18$ or $x=2.$

To check the validity of the two statements,

$$x=18 \text{ implies } \sqrt{3x-5}=1+\sqrt{2x}$$

or

$$x=2 \text{ implies } \sqrt{3x-5}=1+\sqrt{2x},$$

we proceed as follows:

Substitute 18 for x.

Left member is $\sqrt{3(18)-5}=\sqrt{49}=7.$

Right member is $1+\sqrt{2(18)}=1+\sqrt{36}=7.$

Since the left member is the same number as the right member, 18 satisfies the given equation and is a root.

Substitute 2 for x.

Left member is $\sqrt{6-5}=1.$

Right member is $1+\sqrt{4}=3.$

Hence, 2 is not a root of the equation because it does not satisfy the equation.

The first part of the process of solving carries out the proof of the assertion: For any number x,

$$\sqrt{3x-5}=1+\sqrt{2x} \text{ implies } x=18 \text{ or } x=2.$$

One might say that the first part of the procedure finds the only numbers that are eligible to be true substitution instances, and the second part of the procedure, the check, sorts out from the eligible numbers those numbers that are true substitution instances, that is, that satisfy the equation and are therefore roots.

Students sometimes feel that algebra has got out of control in such equations. It has not kept its promise, so to speak. That is because they read more into the process than there is to be read. The algebra has not promised a root and then failed to provide one. When the number 2 fails on the substitution test, we can say that 2 is not a root. Moreover, after we have shown that 18 satisfies the equation and 2 does not, we need look no further—for there are no more possibilities.

Most teachers and almost all intermediate algebras call a number like 2 in the foregoing example an *extraneous root.* We submit that this is a contradiction in terms. From use of the word "root," we infer that we are talking about a number that satisfies an equation. Then how can a number that does not satisfy an equation be any kind of root? It is not the word "extraneous" that we find objectionable but the phrase "extraneous root." We recommend that the phrase "extraneous root" be dropped. Let us be consistent

in our use of the word "root" and agree that a root is a number that satisfies a given equation.

Summary

In this paper we have presented a point of view concerning the importance of "if and only if" or "necessary and sufficient" conditions in secondary school mathematics. More than this has been presented, we believe, because our algebraic examples are done in a form that teachers might well try out. We have found this form helps students in their understanding of what it means to say a number is a root of an equation. In addition, we recommend that the word "check" be carefully used in mathematics classrooms. The phrase "extraneous root" seems to us one that needs change; perhaps "extraneous number" would be better, if a label is desired for a number that arises in the usual process of solution and does not satisfy an equation.

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Have you read?

BRONOWSKI, J., "The Educated Man in 1984," *The Advancement of Science*, Published by British Association for the Advancement of Science, December 1955, pp. 301–306.

Education is always designed to meet the needs of the future. This is a look into that future and we must all concern ourselves with it even though our looking does not show the same picture. What does the author see?

He sees "culture" as Chaucer, Mozart and Science which gives order, unity and intelligibility to the facts of nature. Science not only concerns a vocation but also real living. One cannot be intelligent and have no scientific knowledge.

Since this is true we must start early in education to familiarize the student with science. Mathematics and statistics must come alive. Relationships of all the sciences must be emphasized. Science must be taught as an evolution of knowledge, not a collection of facts, and this goes to the heart of the scientific method. Every one should experience some personal research. It looks like the language of science will be spoken in 1984.

WAGEL, ERNEST and NEWMAN, JAMES R., "Godel's Proof," *Scientific American*, June 1956, pp. 71–87.

This article may be heavy reading for your better students but you should by all means get them to read it. It is an excellent example of the dynamic nature of mathematics and the enormous progress which has been made during the past few years.

Godel, in a paper published in 1931, set forth the conclusion that one cannot find a consistent system of logic that will encompass all of mathematics and that gives any consistent set of axioms that are true mathematical statements which are not derived from the axioms.

These two conclusions of his study changed much of the thinking in modern-day mathematics. The logic back of these conclusions your students may not want to read, but have them read the background material which briefly and clearly points out how we happen to be taking the approach we are in mathematics. This will give both your students and you a renewed appreciation of creative reasoning.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

Proximity of prerequisite learning and success in trigonometry in college

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Educational nomenclature would call this "action research." This is action research at its best.

THIS INVESTIGATION was concerned with discovering whether there is a significant difference in the final grades in trigonometry in college for groups of students differing with respect to the lapse of time since the completion of their last preceding mathematics course.

The records of 805 unselected students who completed the course in trigonometry at Brooklyn College were examined to determine the lapse of time since the completion of their preceding mathematics course, which was generally intermediate algebra. It was found that 172 of these students had completed their preceding course the previous June and thus had a time lapse of only a summer vacation before beginning their study of trigonometry. The remaining 633 students had a minimum lapse of eight months and a mean lapse of seventeen months.

The following table shows the comparison of final grades for these groups.

FINAL GRADES IN TRIGONOMETRY AND LAPSE OF TIME SINCE COMPLETION OF PRECEDING MATHEMATICS COURSE

Final Grade	*"Close Proximity" Group Lapse of summer vacation only* (172 *students*)	*"Greater Lapse" Group Minimum lapse: 8 months Mean lapse: 17 months* (633 *students*)
A	13%	14%
B	22	27
C	24	31
D	22	20
F	19	8
Mean	72.0	76.1

The data in the above table show that the group with greater lapse since the completion of the preceding mathematics course had a failing rate of only 8% in trigonometry as compared with 19% for the close-proximity group. The difference is 3.9 times the standard deviation of the difference between the proportions. Thus the difference between the failing rates for these two groups may be considered statistically significant and not merely due to the chances of sampling.

The difference between the means in favor of the greater-lapse group, i.e., 76.1 as against 72.0, may also be considered statistically significant since it is 3.3 times the standard deviation of the difference between the means.

To obtain data on the comparison of these two groups at the beginning of the course in trigonometry, a pretest was administered to 280 unselected students at the beginning of the term. This pretest contained topics of high school mathematics that find significant application in the study of trigonometry. The raw scores on this pretest were converted into standard scores with a mean of 50 and a standard deviation of 10. The mean scores of the 52 close-proximity students and of the 228 greater-lapse students on this pretest were 54.2 and 49.2 respectively. The difference in favor of the close-proximity group was statistically significant, being 3.2 times the standard deviation of the difference between the means. Thus the close-proximity group would have been considered the more proficient group in background skills at the beginning of the term.

In summary we find that:

1. The students who had completed a mathematics course the immediate semester before studying trigonometry had begun their trigonometry course with greater proficiency in the prerequisite topics than the students with a greater lapse of time since their earlier instruction in mathematics.

2. The students of the greater-lapse group actually earned significantly higher final grades in trigonometry and had a significantly lower failing rate than the close-proximity group.

Because a test administered at the beginning of a course is a static measure of the student's functional abilities and because the loss of knowledge for functional application at the beginning of the term is greater for the greater-lapse group, many superior students of that group score lower than less capable students of the close-proximity group. Such a test does not evaluate the student's dormant knowledge which would again become functional after being refreshed with review.

The failure to obtain decisive results in various experimental studies predicated on the classification of students according to ability as indicated by a pretest may be due to the invalidity of a classification that does not take into account the lapse of time since the student's previous study of the subject.

"To leave a vestige of oneself in the development of another is a touch of immortality. A teacher is a person with a touch of immortality, and he should be the most envied among men." —*President Gould, Antioch College, as quoted by Samuel Lenher in a speech* A New Horizon For Scientific Education *delivered at the National Science Teachers Convention (1956) Washington, D.C.*

Numerical ideas can be normally acquired and numerical operations fully mastered only by arrangements of things—that is, by certain acts of mental construction, which are aided, of course, by acts of physical construction; it is not the mere perception of the things which gives us the idea, but the employing of the things in a constructive way.—*Taken from* The Psychology of Number *by James A. McLennan and John Dewey (New York: D. Appleton and Company, 1916), chap. IV, p. 61.*

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

From ancient China 'til today!

Sun-Tsu, a Chinese of the first century A.D., proposed the problem of finding a number which when divided by 3, 5, and 7 would give the remainders 2, 3, 2 respectively. He gave in verse a rule for this problem which is equivalent to saying that $23+3\cdot5\cdot7\,n$ is a solution for all integral values of n.

This rule did not apparently become known in Europe until after 1852. But in the meantime Nichomachus, a Greek neo-Pythagorean of about 100 A.D., had given the same problem and the answer 23, and the Chinese Yih-hing of the seventh century had generalized the problem. Similar remainder problems had been worked on by the Hindus Aryabhata (c. 500), Brahmagupta (c. 620), Bhaskara (c. 1130), the Arab Ibn Al-Haitam (c. 1000), Leonard of Pisa (1201), and many others.[1]

Our chief purpose here, however, is not to trace the detailed history of the Chinese problem of remainders, but to point out, in more or less chronological order, its connections with a number of other elementary topics. Some of these topics are of current interest because of their possible utility as enrichment for superior students or as simple examples of "different" mathematical systems such as are being called for in some proposals for curriculum revision. In particular, we shall point out relationships to linear equations, congruences in algebra (and so-called finite arithmetics or "arithmetics on a dial"), the Euclidean algorism, and continued fractions. Perhaps some of our readers with interests in these matters will send us further details of the history of the remainder problem, of the theories mentioned above, or of some of the other fanciful and amusing related problems.

Sun-Tsu's problem when written in modern symbolism would be to find an integral value for x satisfying the equations

$$\frac{x}{3}=y+\frac{2}{3}, \quad \frac{x}{5}=z+\frac{3}{5}, \quad \frac{x}{7}=w+\frac{2}{7},$$

or

$$x-3y=2, \quad x-5z=3, \quad x-7w=2$$

where y, z, w are also integers. Each of these is a linear equation which has an infinite number of solutions and which graphs as a straight line until we restrict ourselves to integral values. When we do this to the first of the equations, we see that since $x=3y+2$, every integral value assigned to y will lead to a corresponding integral value for x and this equation has an infinite number of both positive and negative integral solutions. However, if these solutions were graphed they would lie on the line represented by $x-3y=2$ at its so-called lattice points; i.e., those of the points on the line whose co-ordinates are integers.

From graphical considerations one can visualize that there could be linear indeterminate equations with only a finite number of lattice points in the first quadrant and even lines with no lattice points at all in any quadrant. The problem of de-

[1] L. E. Dickson, *History of the Theory of Numbers* (Washington: The Carnegie Institution, 1920), vol. II, pp. v–vi, 57–64.

termining the number and nature of the positive integral solutions of $ax+by=c$ was studied by Paoli in 1780, and it was Euler who gave, in 1760, the simplest proof that this equation is always solvable if a and b are relatively prime.[2] On the basis of this it is easily shown that $ax+by=c$ is solvable in integers if and only if the greatest common divisor of a and b also divides c.

To illustrate a method which works even when the problem does not have a unit coefficient to make it simple, as in the case of $x-3y=2$, let's take a problem given by the sixteenth-century Hindu writer Paramesvara to illustrate a rather obscurely described process given by both Aryabhata and Brahmagupta.[3] The complete problem is to find an integer which when multiplied by 8 and divided by 29 gives a remainder of 4, and which when multiplied by 17 and divided by 45 gives a remainder of 7.

The first condition is expressed by the equation $8x-29y=4$. The solution may be obtained by dividing 29 by 8, 8 by the remainder of this division, etc., until a remainder of 1 is obtained; then, in a sense, reversing the process to find values of x and y such that $8x-29y=1$; and finally multiplying these values by 4, thus:

$$(1)\qquad \begin{array}{llllllll}
\underline{8} & | & 29 & | & \underline{3} & & & \\
 & & \underline{24} & & & & & \\
 & & \underline{5} & | & 8 & | & \underline{1} & \\
 & & & & \underline{5} & & & \\
 & & & & \underline{3} & | & 5 & | \ \underline{1} \\
 & & & & & & \underline{3} & \\
 & & & & & & \underline{2} & | \ 3 \ | \ \underline{1} \\
 & & & & & & & \ \ \underline{2} \\
 & & & & & & & \ \ 1
\end{array}$$

The last remainder may be rewritten in the form

$$(2)\qquad 1=3-1\cdot 2,$$

and the preceding steps as:

$$(3)\qquad \begin{aligned} 2&=5-1\cdot 3\\ 3&=8-1\cdot 5\\ 5&=29-3\cdot 8 \end{aligned}$$

Substituting from (3) into (2) we get:

$$(4)\qquad \begin{aligned} 1&=3-1\cdot(5-1\cdot 3)\\ &=2\cdot 3-1\cdot 5\\ &=2(8-1\cdot 5)-1\cdot 5\\ &=2\cdot 8-3\cdot 5\\ &=2\cdot 8-3(29-3\cdot 8)\\ &=11\cdot 8-3\cdot 29. \end{aligned}$$

Thus $x=11$, $y=3$ satisfies $8x-29y=1$, and $x=44$, $y=12$ will satisfy $8x-29y=4$.

The process displayed in step (1) is also the Euclidean algorism for finding the greatest common divisor of 8 and 29 (see Euclid's *Elements*, Book VII, Proposition 2, for its geometric origin and any theory of equations or higher algebra book for its algebraic proof and use). The fact that this g.c.d. is 1 means both that 8 and 29 are relatively prime and that the substitution process (4) which gave the solution of $8x-29y=1$ will be possible.

Of course, 44 and 12 are not the only integral solutions of our equation. It can be shown that if (x_0, y_0) is an integral solution of $ax+by=c$ then $x=x_0+bt$, $y=y_0-at$ is a solution for all integral values of t.[4] Thus, for our example $x=44-29t$, $y=12-8t$, and for $t=1$, we have $x=15$, $y=4$. This last is the result originally obtained by Paramesvara by using a "chain" described rather vaguely by both Aryabhata and Brahmagupta. Öre gives an algorism for constructing a chain which is essentially the same as the Hindu chain, but easier to set up and explain.[5] His chain is constructed by writing

[2] *Loc. cit.*

[3] Walter Eugene Clark, *The Aryabhatiya of Aryabhata* (Chicago: University of Chicago Press, 1930), p. 44 ff. See also Henry Thomas Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmegupta and Bhaskara* (London: 1817), p. 325 ff.

[4] Oysteen Öre, *Number Theory and Its History* (New York: McGraw-Hill Book Co., 1948), p. 149.

[5] Öre, *op. cit.*, pp. 142–149.

the quotients from the algorism (1) as in the center, or q_i, column here and then constructing the chain in a parallel c_i column by arbitrarily writing 1 and 0 in the c_i column below the level of the last q_i and then filling in the c_is *from the bottom up* by multiplying each q_i times the c_{i+1} on the line below it, then adding the c_{i+2} on the second line below it; i.e., $c_i = q_i c_{i+1} + c_{i+2}$. The two top links in the chain, 11, 3, are a solution of the equation $8x - 29y = 1$. These produce a solution of the original equation when multiplied by 4.

i	q_i	c_i
1	3	11
2	1	3
3	1	2
4	1	1
5		1
6		0

Öre's chain can be derived as a general rule by generalizing the g.c.d. process and the "reverse" substitutions which we used in our first solution. Paramesvara's interpretation of Aryabhata's process begins with the g.c.d. division process and constructs a chain by the same rule as Öre's except that by properly choosing two numbers different from the 0 and 1 with which we began our second column he ends with a solution of $8x - 29y = 4$ immediately.

We leave it to the reader to complete the solution of the original problem, and instead note that the g.c.d. process could have been replaced by the conversion of the fraction 8/29 into a simple continued fraction thus:

$$(5) \qquad \frac{8}{29} = \frac{1}{\frac{29}{8}} = \frac{1}{3+\frac{5}{8}} = \frac{1}{3+\frac{1}{\frac{8}{5}}} = \left\{ \frac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}} \right.$$

The convergents of the final continued fraction are 1/3, 1/4, 2/7, 3/11, 8/29. Note that the denominator and numerator of the next to the last convergent are again our solution, (11, 3) of $8x - 29y = 4$.

This continued-fraction approach will work on all linear indeterminate equations in two unknowns. Such equations are one type of Diophantine equation, indeterminate equations for which all their integral solutions are sought. These are named after the Greek algebraist Diophantus, who lived in the first century after Christ. However, Diophantus himself left no writings on linear indeterminate equations, although he did work with second degree and other "Diophantine equations." But in general he was satisfied with finding a single solution where later mathematicians have sought to find all of the solutions.

Although some idea of continued fractions has been read into this work of Aryabhata's and into the early Greek treatment of quadratic irrationalities, it seems an exaggeration to attribute to them such a general and theoretical knowledge of continued fractions as makes it possible to say that the penultimate convergent of such a fraction will always give a solution of the corresponding linear indeterminate equation. Italian Pietro Antonio Cataldi (1548–1676) and Swiss Leonard Euler (1737) were chief among the early workers who had a real conception of

continued fractions.[6]

Finally, we note that C. F. Gauss in his *Disquisitiones Arithmeticae* of 1801 defined two integers a and b to be *congruent* modulo m when their difference is divisible by m. That is, $a \equiv b \bmod m$ if and only if $(a-b)=my$ or $my+b=a$ for some integer y. Hence the congruence $x \equiv b \bmod m$ can be written $my+b=x$ and the solution of the congruence becomes equivalent to the solution of a linear indeterminate equation. Thus linear equations, their graphs and the related lattices are related to "modular arithmetics" or "arithmetics on dials," and in the related algebra of congruences some linear "equations" (congruences) may have no (integral) solutions or a finite or infinite number of positive (or negative) solutions. Modular arithmetics and congruences furnish rather simple and elementary illustrations of the existence of different but comparable structures in algebra, just as non-Euclidean geometries illustrate the existence of different structures in geometry. Modular arithmetics and congruences also demonstrate the nature and role of definitions and axioms in the foundations of mathematics.

[6] See H. S. Hall and S. R. Knight, *Higher Algebra* (London: Macmillan, 1932), pp. 111–112, Chaps. XXV, XXVI for a simple explanation of continued fractions and indeterminate equations. For historical comments and references see E. T. Bell, *The Development of Mathematics* (New York: McGraw-Hill Book Co., 1940), pp. 277, 444, and D. E. Smith, *History of Mathematics* (Boston: Ginn and Co., 1923), vol. I, pp. 156, 303.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Calculus, Jack R. Britton, New York, Rinehart and Co., 1956. Cloth, xiv+584 pp., $6.50.

This is a text for a standard college course in calculus. The influence of the recent cries for more emphasis on the concepts and logical aspects of calculus is seen only in the first fifty pages which are devoted to an earnest effort to explain the limit concept in terms of the ancient definition of Weierstrass. The lack of a suitable geometric picture to accompany this arithmetic view of limits has made its use, even in some current research in analysis, rather unpopular. There seems to be little doubt that this approach is ineffective for most undergraduate students. The geometric view of limits as embodied in the "filters" of H. Cartan (*alias* N. Bourbaki) shows some promise of providing a solution for this vexing problem of pedagogy.

The remainder of this text is devoted to an exceedingly large number of topics directed toward the use of calculus in engineering applications. Statistical and other applications are ignored, except for a few exercises. Even if the imaginative teacher of calculus is not repelled by this mundane view of his subject, he will scarcely be happy with the treatment accorded many of the purely mathematical topics. A list of instances in which the author has departed from well-established definitions or has given quasi-proofs is too long to be included here. Let it suffice to observe that two favorite words of the author seem to be "intuitive" and "plausible."

The text is replete with exercises, both illustrative and formal. Each chapter closes with a list of questions for review and discussion. These are, of course, excellent features of any text.—*M. F. Smiley, State University of Iowa, Iowa City, Iowa*

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Junior high school mathematics and the manpower shortage

by Clyde E. Parrish, Cubberly High School, Palo Alto, California

Introduction

At the present time in the United States a shortage of scientific, professional, and technical manpower exists. The problem of alleviating this shortage belongs to American society in general, and in particular it belongs to the educational agencies of our society.

As a fundamental in the education of scientific, professional, and technical persons, mathematics education becomes an area of further focus for the problem created by this shortage. The junior high school has functioned as a level of education that places considerable emphasis on guidance in relation to the student's present and future. Some educators have identified the junior high school years as a crossroads period. What the student does in these years has great import as far as career choices are concerned. Patterns of behavior will be formed that will either permit an unrestricted or a restricted career choice when the student feels moved to make that choice.

What the junior high school is doing and what it is not doing to influence these patterns of behavior in relation to mathematics is the primary concern of this paper. Such a consideration requires an analysis of secondary school mathematics, past and present, since it may be one of the factors contributing to the existence of the current shortage.

Analysis of secondary school mathematics

A recent report on the subject of mathematics education considered three major factors, the student, the teacher, and the curriculum.[1] To these factors I would add one more: this fourth factor is our society. Of these four factors I think that the society and the curriculum need the most extensive and critical consideration.

The Society

In the final analysis our social organization, in terms of its economic, political, and technological factors, determines the necessary education of any individual preparing to play a role in society. This demand has been given extensive consideration in the professional education of mathematics teachers as evidenced by discussions found in textbooks designed for the prospective mathematics teacher.[2] The professionally trained mathematics

[1] Henry S. Dyer, Robert Kalin, and Frederic M. Lord, *Problems in Mathematical Education* (Princeton, New Jersey: Educational Testing Service, 1956).

[2] Charles H. Butler and F. Lynwood Wren, *The Teaching of Secondary Mathematics* (New York: McGraw-Hill Book Co., 1951); Lucien B. Kinney and Richard C. Purdy, *Teaching Mathematics in the Secondary School* (New York: Rinehart, 1952).

teacher, therefore, is aware of the needs of specific roles played by individuals in our society.

However, this awareness on the part of mathematics teachers is apparently not accompanied by a like awareness on the part of the great bulk of our population. The old familiar refrain from parents, "I had algebra in high school and haven't used it since," is testimony to this lack of awareness. I have even heard a remark from an engineer that indicates that engineers may not recognize the significance of mathematics in their day-to-day work activities. In answer to a query from a junior high school mathematics teacher, I heard an engineer say, "The only value mathematics has had for me was to provide mental discipline." Yet, a few minutes before this remark, he had pulled a slide rule from his breast pocket and had done a quick computation in relation to something he was thinking about. This engineer was so habituated in his usage of mathematics that he was overlooking his rather obvious, probable daily, usage of mathematics in a great many different ways.

This lack of awareness of the significance of mathematics on the part of the general public poses a real problem in terms of selling the value and importance of mathematics to members of our society. It also raises the possibility that there is a lack in the curricular organization of mathematics that is contributing to this low level of awareness.

The Curriculum

As was pointed out previously, extensive consideration has been given to the mathematical needs of individuals as created by the demands of modern society. These needs have been a strong governing factor in determining the advocated curriculum for secondary school mathematics. However, a stronger factor has been in operation and in opposition; it has a hold, through tradition, that is hard to break. It is the college preparatory character of high school mathematics courses. In some respects the preparation for a life in society and life in the intellectual climate of a college happen to coincide. Both call for the development of computational skill and the development of understanding of concepts. The real disagreement created by the objectives of preparing for two types of life activity seems to lie in how the objectives shall be reached rather than in differences in the actual objectives. There has been considerable preoccupation with this latter problem, possibly to the exclusion of other curricular considerations that may be more important.

My purpose is to avoid this conflict and give consideration to at least one other facet of the mathematics curriculum. This other facet is the objectives themselves.

Butler and Wren summarize these objectives as follows:

1. Proficiency in fundamental skills,
2. Comprehension of basic concepts,
3. Appreciation of significant meanings,
4. Development of desirable attitudes,
5. Efficiency in making sound applications,
6. Confidence in making intelligent and independent interpretations.[3]

Other writers have made similar expressions of the objectives of secondary mathematics teaching. I will assume these to be representative.

Examination of published material and the collection of data from experience reveals an emphasis upon basic skills associated with computation as related to the practical considerations of life so narrowly focused that it has made some of the other objectives inoperative. As evidence of this, consider the emphasis represented by the *Guidance Pamphlet in Mathematics for High School Students*.[4] Its primary emphasis is upon mathematical competencies necessary for effective citizenship

[3] Butler and Wren, *op. cit.*, p. 16.

[4] By Commission on Post-War Plans of the National Council of Teachers of Mathematics (New York: THE MATHEMATICS TEACHER, 1947).

and for satisfactory performance in certain vocational and professional roles.

Personal experience reinforces this evidence. Several years ago a particular school system asked each secondary school teacher to prepare a statement of objectives for each subject that he was currently teaching. Without exception, the teacher of mathematics who taught only mathematics turned in a series of objectives that were pure subject matter in orientation. The statements, in some instances, were an outright copy of the table of contents of the textbook in use or a close approximation of it. No mention was made of the development of attitudes, of the encouragement of interest, or of an appreciation of mathematics as teaching objectives. The teaching that these people do may have an implicit subscription to these objectives, but it is certainly not an explicit, conscious subscription to them. It is probable that this condition is not confined to this school alone.

It is important to recognize, at this point, that the active operating curriculum will be based on the objectives to which mathematics teachers have explicitly subscribed. If this is true, and the evidence presented is valid, our present day mathematics curriculum is much too narrow. The following is an indication that this is the real condition:

> An increased and more accurate understanding of the learning process, which has resulted from the scientific study of pupils, explains the ineffectiveness of a great deal of past and present day teaching of mathematics. It calls attention to the importance of attitudes, understandings, and interests as well as manipulative skills.[5]

At this point it is rather obvious that an hypothesis needs formulation and testing. This part of our manpower shortage can be corrected by teaching mathematics so as to induce an attitude of appreciation, a stimulation of interest, and an understanding of the part played by mathematics in our culture.

The Teacher

At least two kinds of mathematics teachers are presently teaching in our public secondary schools. One kind is the person who wanted to be a mathematics teacher because of an interest in mathematics combined with an interest in young people. The second kind of mathematics teacher has the interest in young people in common with the first type of teacher, but he is teaching mathematics not out of interest or choice, but because of the school and the programming problems peculiar to it. Oddly enough in the appraisal of pupils and staff, the second type of mathematics teacher is often considered a better teacher of mathematics than the first type.

Another common factor in these two types of mathematics teachers is their student experience with mathematics. Both took, and are now probably teaching, computationally oriented courses, and if any of their students become teachers of high school mathematics they will also probably teach computationally oriented courses, unless something is done to break the cycle.

The first type of teacher, since he is teaching mathematics voluntarily, probably finds considerable intrinsic motivation in the subject material of mathematics. Because he has this feeling for mathematics, he may be somewhat annoyed by the fact that not all of his students see the subject matter in the same light that he does. He may as a result concentrate, and apparently often does, on the students who share his feeling for mathematics while the others struggle along as best they can with minimum attention.

This circumstance of the dubious influence of the teacher, according to the evidence,[6] is occurring all too often, and quite decidedly, it must be concluded, to the detriment of a general appreciation for mathematics and interest in it.

[5] Kinney and Purdy, *op. cit.*, p. 38.

[6] Dyer, Kalin, and Lord, *op. cit.*, p. 15.

On the other hand, the second type of teacher may have a decidedly sympathetic feeling for students who are having difficulty with mathematics. This would be true particularly if he had had similar troubles with mathematics during his student days. Potentially, more hope can be attached to the efforts of the second type of teacher if he can struggle through to a point of appreciating mathematics as an important part of modern society. He is more likely to *teach* the greater bulk of his students than the teacher who is motivated by his own intrinsic interest in mathematics. Yet to attach hopes of solving the problem of a manpower shortage to such a thin possibility would be unwise.

Both types of mathematics teachers need to expand their perception of mathematics and their perception of mathematics teaching. One of these perceptions is to realize the existence of, and the need for, subscription to a broader set of teaching objectives. The curriculum cannot expand without this expanded perception on the part of the teacher.

The Student

At the very center of this consideration of mathematics stands the fellow who is to learn, the fellow who is subject to the teaching of the teacher. Modern educational psychology labels the student as the most important consideration in the school process. I am inclined to conclude that mathematics teachers, and therefore the mathematics curriculum, have not yet taken adequate account of this point of view. They have been exposed to it, but as yet have not worked out how to apply it in mathematics teaching.

One of the basic postulates associated with modern views of learning is that learning readily occurs only when proper motivation is present. Mathematics teachers have attempted to follow this postulate to a certain degree. It has led to the previously mentioned wide consideration of the individual's mathematical needs as dictated by social needs. But this has established only what the student needs and not "why" he needs it nor how to incorporate it into the student's emotional structure.

The student may be told that he needs to know how to compute the tax on his income when he becomes the earner of an income. He, knowing something of the ways of government, accepts this as fact and agrees with you, but he is not sufficiently aroused emotionally by this fact to undertake a calculation for a hypothetical income. And why should he become aroused? No normal adult waxes enthusiastic over calculating how much Uncle Sam is taking away this year.

This raises the question, "Can we appeal to the emotional system of the student and get him to internalize these apparent future needs through a curriculum based and organized directly on his future needs?" (This is intended to include college preparatory needs.) Evidence existing at the present time seems to indicate a negative answer.

While we search for a more satisfactory means of motivation we exercise our teacher authority to force students to go through the motions. They may absorb some mathematics through this practice, but it is doubtful whether many will fully appreciate what they have absorbed. Until we find a satisfactory means of motivating mathematics students, we can only mark time in relation to solving the problem of a manpower shortage. In other words, the problem of arousing genuine interest in mathematics is just as important as the educational psychologist says it is. Under the present high school mathematics curriculum there is little possibility of achieving the needed motivation.

Summary and Recommendations

The following points are central in the above analysis:

A. In relation to society:
   1. There is a general lack of awareness on the part of the general public of the multitude of ways in which mathematics is used by all people in their daily living.

2. This lack of awareness may be due to the nature of the mathematics curriculum.

B. In relation to the curriculum:
   1. The explicit objectives of secondary school mathematics seem to center around computational skills, basic concepts, and understandings.
   2. The focus on skills, concepts, and understandings seems so complete that interests, attitudes, and appreciations are excluded as objectives in the explicit sense.

C. In relation to the teacher:
   1. Mathematics teachers will teach effectively only toward objectives that they explicitly recognize and accept.
   2. Teachers of mathematics tend to teach as they were taught, ad infinitum. The cycle needs to be broken.
   3. The mathematics teacher, therefore, needs to be educated to a broader perception of the significance of mathematics.

D. In relation to the student:
   1. Through the way it is taught, mathematics has appeal to a rather limited group of students.
   2. Mathematics is an area where motivational problems of the first magnitude exist.
   3. We might better solve some of the problems of motivation by an indirect, flanking approach to future needs rather than by continued frontal attack.

An appropriate question at this point is: What implications and conclusions can be drawn from the line of reasoning followed in this paper?

The first conclusion is that the nature of the mathematics curriculum, past and present, is directly related to the lack of awareness and the lack of appreciation that the general public feels in regard to mathematics. The American value system places mathematics at a rather low level. This indifference to objectives concerning attitudes, interests, and appreciations is the specific portion of the curriculum or lack of curriculum that is creating this lack of awareness and lack of appreciation. Something that is not specifically taught cannot be learned in other than a very incidental and accidental manner.

A second conclusion is that the persistent emphasis upon computation, i.e., the student being a comptometer, tends to destroy interest rather than encourage it in many students. This seems much like an English teacher who insists on a thirty-six week diet of grammar unleavened by excursions into literature. She would be faced by rebellion before many weeks. Many mathematics students do rebel at the deadly computational diet and manifest their rebellion as indifference. What is needed is a "literature" for mathematics, an anthology of stories, poems, and anecdotes that would utilize the language of mathematics in the telling and also help to reinforce the basic skills and concepts that we are currently trying to teach. Some effort has been directed toward this development, such as the writings of David Eugene Smith. Others have tried to popularize mathematics, but their writings do not lend themselves readily to integration into a curricular development.

A third conclusion points to something missing in the present secondary school curriculum. This is a treatment of mathematics with appropriate illustrations as one of mankind's proudest achievements. The invention or discovery of even the simplest number relations are achievements on a par with the invention of the telephone, the telegraph, and the internal combustion engine. Why aren't they presented and studied as such, as well as being presented as useful skills?

Much school time is devoted to political history which is characterized by the human effort to dominate one another. Mathematics can be looked upon, historically, as man's effort to conquer abstract ideas and put them to use. Why is this history not more important than political history? It has contributed more to the welfare and progress of mankind than political history. Why isn't it taught as part of the secondary school mathematics program? The answer probably lies in the fact that little really critical consideration of what they are doing is being displayed by mathematics teachers, who teach as they were taught, in a curricular dead end paved with computational bricks. We need to back out of this dead end and start over.

A further conclusion grows out of the previous conclusions. It is to the effect that motivational difficulties can be overcome by a broadening of the scope of the curriculum along the lines suggested by the aforementioned conclusions. Mathematics, centered around computation alone due to its narrow appeal is a denial of individual differences; expansion of that appeal would work its own cure on some of the motivational difficulties that exist at present.

This series of conclusions leads to a second question that is most appropriate. What courses of action can be recommended, assuming the line of reasoning of this paper to be correct?

One course of action might be to produce the mathematical anthology previously mentioned. Perhaps one of the philanthropic foundations would be interested in supporting such a project. Very little short of a team composed of a mathematics educator, a children's writer, and a mathematician would be required to compose and compile such a book. For individual teachers to undertake such collecting and such writing would be placing this possibility in the dead file.

Another course of action would be to arrange teacher workshops for two purposes. One would be to educate teachers as to the cultural significance of mathematics and the second would be to consider operational definitions of the desired attitudes, interests, and understandings that are presently inoperative objectives of mathematics education. Defining objectives in behavioral terms would also help lead into techniques for achieving the desired ends. It has been my experience that group effort on a project of this kind leads to more fruitful results than individual effort. Too often the individual is at a loss as to where to start; and we need to start. Perhaps, again, there might be private interests who would underwrite the cost of such a project. It is being done in other educational fields.

A third course of action might be to aim at the adult public information other than statistics that emphasize the extent of our manpower problem. Newspapers and popular magazines might be quite willing to print articles if someone were motivated to produce them.

Conclusion

There has been an underlying thesis implicit in this paper that the reader may have detected by this time. He would be most likely to detect it if he were quite familiar with the postulates of educational psychology. Most illustrations of this postulate are in terms of puzzle solving or in relation to practice effects. It is the predisposition to think along certain lines and erect in a problem restrictions that do not actually exist. Some people are good puzzle solvers because they habitually can avoid this set or predisposition; others never seem to be able to avoid these false and unnecessary assumptions.

One would think that mathematics teachers would tend to fall among the group who habitually avoid a restrictive set. However, what this paper has pointed out indicates that, at least in the instance of curriculum, mathematics teachers have been very subject to a particular set. They have assumed that the mathematics curriculum is restricted to activities closely related to computation as reflected in modern day textbooks for mathematics courses. This set has led to a dangerous state of affairs marked by a distinct decline in interest in mathematics on the part of the American populace. This, in turn, has contributed to our current manpower shortage.

We mathematics teachers have interpreted our mandate from society too narrowly; now we need to do something about it. As Lucien Kinney and Richard Purdy say, "This is a challenging responsibility. We cannot stand still."[7]

[7] *Op. cit.*, p. 28.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Careers for the mathematician . . .

There has been much ado these past few years anent the manpower shortage in science and mathematics. This despite the fact that the literature of vocational counseling has been more than moderately abundant, including the pioneer *Guidance Pamphlet* first issued by the National Council of Teachers of Mathematics in 1947, as well as popular pamphlets issued by the Canadian Mathematical Congress, the Mathematical Association of America, and the General Electric Company,[1] to mention outstanding publications.

A more recent pamphlet[2] adopts a somewhat different approach—the direct viewpoint of the prospective employer of young people trained in mathematics. This 32-page document is quite practical, and includes a useful employers directory. We quote by permission, from an editorial,

MATHEMATICAL POSSIBILITIES: THE OUTLOOK.

The neophyte mathematician has five general categories into which he can project himself: teaching, statistics, industrial research, government and actuarial work.

CAP AND GOWN

Teaching mathematics demands special qualities. One may find himself explaining the differences between geometric systems in the morning, and later, in some schools, the advantages of the single-wing over the T-formation. He must have the personality to overcome adolescent prejudice against his field, or perhaps to convince graduate students that mathematics can be thought of as a philosophy. He must not only have the history of the field within his grasp, but must also, at the university level, find time for productive research. It is also imperative to integrate his program with the other sciences or arts according to the policy of the institution.

EXPANDING FIELD

Mathematical and applied statistics is a field that has had a renaissance in the last quarter century. Modern marketing methods, advertising techniques, and sampling schedules have made the trained statistician a scarce commodity. Sociologists, Psychologists and Public Opinion Researchers need statistically trained personnel to deal with general theory, sift hypotheses, and formulate ways to test them through experimentation. Competition has forced industry to maintain rigid quality control in manufacturing, and also to feel that the cost of canvassing the public to determine their preferences and purchasing habits is an imperative investment. This is a mushrooming field with numerous opportunities for both men and women.

DEMAND FOR DOCTORS

The key to opportunity for Mathematicians in industry is the Ph.D. degree. Those with lesser degrees will normally begin in some computing activity and progress from there. The prospective industrial mathematician should be conversant with the language of his engineering colleagues, and, ideally, should diversify his interests into study in the physical sciences. He will be expected to supply solutions to specific problems inherent in aero-elasticity, for example, in gravimetry or turbine instability, depending on the field of his choice. In research, however, the solutions may be determined by any part of the mathematical spectrum from algebra and trigonometry to Fourier and Laplace transforms.

SUBMARINES AND SUBWAYS

The Government, along with the definite assets and liabilities of Civil Service, provides opportunities for both men and women at all levels of mathematical proficiency from the junior to the professional. In the lower grades the mathematician will probably be concerned with numerical solutions to differential equations. On the higher levels, as in industry, conversance

[1] *Why Study Mathematics?* Canadian Mathematical Congress (Montreal, Canada: McGill University), 50¢; "Professional Opportunities in Mathematics" *American Mathematical Monthly*, LVIII (January 1951); *Three Why's*; General Electric Company.

[2] *Careers for the Mathematician:* Monograph No. 5; Career Publications, Inc., 14 West 45 St., N. Y. C. (1956); $1.00.

with related scientific fields is demanded, but can be acquired by additional study. In these positions a Ph.D., plus practical experience in relating mathematical solutions to scientific problems are practically essentials. The kinds of problems with which the government must cope vary from logistical military entanglements to the practicality of using various coins in subway turnstiles.

WANTED: YOUNG ACTUARIES

Actuaries are in great demand . . . well paid . . . key figures in industry. Essentially, the actuary is a business man rather than a mathematician. The actuarial profession offers a man with mathematical aptitude a unique opportunity to use this aptitude in the world of business. Technical knowledge, blended with a broad cultural, social, and business knowledge, lead inevitably to policy-making, responsible executive posts in the insurance business. Outside of insurance companies, many actuaries practice the profession as consultants. Others serving in government service are concerned with pension funds, the regulation of insurance and social security, and a variety of demographic problems. Professional standing as an actuary is achieved by attaining membership in the Society of Actuaries; to become a fellow of the Society one must pass a series of eight examinations. The best background is a college education in the liberal arts with a major in mathematics. As reflected by the examinations, the standards for the profession are high.

Some reflections on quadratic equations . . .

In the same year that the British captured Fort Duquesne and renamed it Fort Pitt (now Pittsburgh), the English amateur mathematician Francis Maseres published his treatise on the use of the negative sign in algebra. Maseres also published, at his own expense, a considerable number of reprints of some of the earlier works of seventeenth-century mathematicians, including Napier, Snell, Descartes, Schooten, Huygens, Barrow, and Halley. In 1783, he wrote a substantial treatise on the theory of life assurance, one of the earliest attempts at putting the subject on a mathematical basis.

Although a member of the Royal Society, Maseres contributed but few papers; such as he did write were concerned with the nature of equations, infinite series, logarithms, and optics. Of peculiar interest are his book on trigonometry and several tracts on algebra; they are virtually worthless, since he rejected the use of negative numbers as "impossible quantities." We allude to his work here chiefly to emphasize the amazing resistance which had to be overcome before negative numbers were universally accepted—a protracted struggle lasting several centuries. Here, for example, in 1758, was a writer who reluctantly allowed the use of negative numbers, even though Descartes had definitely established their status in 1637, and the first edition of Newton's *Principia* had appeared fifty years later. He was not the last to resist, however, for in 1796, William Frend, father-in-law of Augustus De Morgan, also published a work on algebra in which he regarded negative quantities as nonsensical.

The Preface to the *Dissertation on the Use of the Negative Sign in Algebra* opens with the following curious observation:

> The design of the following dissertation is to remove from some of the less abstruse parts of algebra, the difficulties that have arisen therein from the too extensive use of the Negative Sign, and to explain them, without considering the Negative Sign in any other light than as the mark of the subtraction of a lesser quantity from a greater.

Elsewhere in this unbelievable tract we read:

> Now all affected quadratic equations are evidently reducible to one of these three forms:
>
> $$xx+px=r$$
> $$xx-px=r$$
> $$px-xx=r$$
>
> . . . if any question or problem produces an equation of either of the forms $xx+px=r$, or $xx-px=r$, it will admit of only one answer; but if it produces an equation of the form $px-xx=r$, it will admit of two answers, unless there be some condition in the problem, which cannot be expressed in the equation, and which determines x to have but one value. . . . Those who explain quadratic equations by negative roots say, that every form of quadratic equations, as well as the third, has two roots; that the third, indeed, is the only one that has two affirmative roots; but that the first and second have each of them an affirmative and a negative root. . . . This seems to the two equations $xx+px=r$, and $xx-px=r$ (which are so many assertions quite distinct from each other, and cannot be derived from the conditions of the same problem), as if they

A

DISSERTATION

On the USE of the

NEGATIVE SIGN

IN

ALGEBRA:

Containing a DEMONSTRATION of

The RULES ufually given concerning it;

AND SHEWING

How QUADRATIC and CUBIC EQUATIONS may be explained, without the Confideration of NEGATIVE ROOTS.

To which is added, as an APPENDIX,

Mr. MACHIN's QUADRATURE of the CIRCLE.

By FRANCIS MASERES, M. A.
Fellow of CLARE-HALL, CAMBRIDGE.

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Printed by SAMUEL RICHARDSON;
And Sold by THOMAS PAYNE, in Caftle-Street, near the Mews-Gate.
M DCC LVIII.

> were one and the same equation. . . . This method of uniting together two different equations may perhaps have its uses; but, I must confess, I cannot see them: on the contrary, it should seem that perspicuity and accuracy require, that two equations, or propositions, that are in their nature different from each other, and are the results of different conditions and suppositions, should be carefully distinguished from each other, and treated of separately, each by itself, as it comes under consideration.

By far the greater part of the volume, which runs to nearly three hundred octavo pages, is devoted to a similar treatment of the nature and solution of cubic equations. The discussion is verbose and labored; the demonstrations make frequent use of fluxions. Limitations of space forbid our going into further details, but enough has probably been said to show the general tenor of the exposition. We confess to amazement at the dogged persistence of the author in tracing all the "types" of cubics, carefully establishing numerical limits (in terms of the arbitrary coefficients p, q, and r) for which there are one, two, or three roots! Not the least remarkable passage perhaps, is the following, which reveals his general philosophy:

> Having . . . considered what is meant by making use of negative roots . . . 'tis natural, before we quit this subject, to observe, that notwithstanding most of the modern writers of algebra have thought fit (probably that they might save themselves the trouble of demonstrating each case . . . separately) to make use of negative roots, yet the resolution of quadratic equations is performed full as concisely, and in a much more simple and natural manner, without than with them; nay 'tis, as I apprehend, necessary to know how to resolve equations without them, before we can understand how they can be resolved by them . . . they serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceeding plain and simple. . . . It were to be wished therefore that negative roots had never been admitted into algebra, or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many learned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed; it being certain that algebra, or universal arithmetic, is, in its own nature, a science no less simple, clear, and capable of demonstration, than geometry.

In addition to the treatise referred to above, Maseres also published *Tracts on the Resolution of Cubick and Biquadratick Equations* (1803), and *Tracts on the Resolution of Affected Algebraick Equations by Dr. Halley's, Mr. Raphson's and Sir Isaac Newton's Methods of Approximations* (1800). The very last section in the latter volume is a brief essay by William Frend, Fellow of Jesus College, Cambridge, entitled "Remarks on the Number of Negative and Impossible Roots in Algebraick Equations" (pp. 473–479). In closing, we condense, in part, one paragraph:

> The supposition which eminent writers on Algebra lay down as an indisputable and almost self-evident maxim is that every Algebraick equation has as many roots as it has dimensions. In truth, there are very few equations in which this maxim really takes place; to wit, only one single form in each degree, or order, of equations. Thus, of the three forms of affected quadratick equations, only the form $px - xx = q$ can ever have two roots, and *that* only when q is less

than $p^2/4$. Similarly, of all the thirteen different forms of affected cubick equations, only the form $x^3-px^2+qx=r$ can ever have three roots, and *that* only when r is less than $p^3/27$. Finally, of the forty-five different forms of affected biquadratick equations, it is only the form

$$-x^4+px^3-qx^2+rx=s$$

that can ever have four roots, and *that* only when s is less than $p^4/256$. . . . Therefore, when these eminent writers have laid down the foregoing general proposition, they find themselves under a necessity of giving specious names to a parcell of quantities which they endeavor to make pass for roots of these equations, though in truth they are not so, in order to cover the falsehood of their general proposition, and give it, in words at least, an appearance of truth; and with this view they call some of these quantities *negative roots* of the equation to which they relate, and others of them it's *impossible roots*.

To be sure, we have come a long way in the development of algebra in the two centuries that have passed since the time of Maseres, Maclaurin, De Morgan, and Frend. Yet a searching study of the quadratic equation—the properties and relations of its roots and coefficients—can readily open new vistas to the beginning student of elementary algebra. With the thought that a full discussion of quadratic equations and related topics might be of interest to brighter pupils, to mathematics clubs, and to beginning teachers, the following bibliography is included as a source of background and enrichment material.

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What's new?

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• POINTS AND VIEWPOINTS

A column of unofficial comment

A new day?

by Maurice L. Hartung, University of Chicago, Chicago, Illinois

Is a new day in mathematical education about to dawn? Will the ferment working in mathematical and science education yield a richer and more stimulating curriculum?

Several groups including committees sponsored by national organizations are exploring ways of improving instruction in mathematics. Institutes designed to help high school mathematics teachers become better grounded in modern mathematical theory are increasing in number. "Experimental curriculums" for high school students are attracting attention and comment which, if not always favorable, is at least not generally hostile.

This is not the first time in educational history that a movement for reform in mathematical instruction has welled up. In the past such movements have received their impetus from two disparate sources. One source was a small group within the mathematical family—notably men like Felix Klein and E. H. Moore. Other mathematicians listened but took no active role in producing changes in the curriculum. The other impetus came from outside the family—primarily from principals and superintendents of schools and professors in schools of education. Mathematicians resented and generally resisted proposals of reform from this source.

At present mathematicians are again taking an active role by demanding changes and participating in the exploration of the specific directions that these changes should take. In contrast to times past, however, there seems now to be a much wider base for support of the movement both within and outside the ranks of professional mathematicians. The situation looks very promising, but this promise may never become reality, or be so long delayed that needs will have changed.

The problem of building an effective curriculum is not a simple one. Although one can find frequent references to "curriculum theory" in educational literature, at the present time the theories that exist must be regarded as rather primitive. Nevertheless, the major elements of a model for curriculum building are widely accepted by professional curriculum workers. The issue to be considered in the paragraphs that follow is this: To what extent does current activity looking toward changes in the mathematics curriculum constitute a realization of modern general curriculum theory?

First, general curriculum theory recognizes that the process of education takes place in situations that involve persons in particular times, places, and circumstances. Among these circumstances is the *content* with which the learner deals. The purpose of education is to change the behavior of these persons; that is, the way they think, feel, and act in given situa-

tions. Statements of the objectives of education are specifications of the behavior that is desired or to be sought for.

Second, the determination of valid objectives depends upon the prior determination of certain other components. One of these components is called a "philosophy of education." If properly formulated it consists of a set of statements of value positions held by those who are determining what the curriculum is to be. For example, is the curriculum to be for all normal youth, or only for a selected subgroup? Are methods and practices to be democratic or authoritarian? Statements of this sort play a role that is analogous to that of the postulates in a deductive logical system.

A second major component of the machinery for curriculum building is a "psychology of learning." This consists of statements of principles of learning theory that are to be governing, and of selected findings of psychological research that are relevant to the decision-making process in curriculum building. In theory, some of these statements play roles not unlike the operational rules in mathematical theory—they tell what you should or should not do in given situations.

When subject-matter specialists, such as mathematicians, start out to build a curriculum they are inclined to treat the philosophical and psychological components in cavalier fashion. The highly trained subject-matter specialist is properly an enthusiast for his field of competence. In curriculum planning he moves as fast as he can toward the identification of the content he believes students should learn. It is true that completion of this task is a necessary condition for the determination of valid objectives, but it is not sufficient. Of at least equal importance are tasks of another sort. Decisions must be made which to be valid depend upon competence in fields other than mathematics. Teachers and professional mathematicians are rarely experts in these fields.

Modern psychological theory, for example, commonly employs selected technical concepts of its own such as *need*, *readiness*, *capacity*, and *inhibition*. Extensive bodies of empirical evidence now exist to support conclusions about learning expressed in terms of these concepts. Time and energy are wasted when efforts to select content and to organize it for instructional purposes proceed without adequate use of philosophical and psychological controls on these processes.

The derivative of a function is a concept of unquestioned importance in mathematics and science, but no sensible person would propose that it be taught in the elementary grades. This is because persons with sufficient intelligence and education to know what a derivative is immediately recognize that elementary school pupils do not have need, readiness, capacity, or motivation for meaningful learning of this concept. To force them would doubtless lead to inhibitions relative to learning other mathematical concepts. In the case of many concepts and situations, the decision to select or reject is not so easily made. In these cases the controls or criteria of the kinds found in curriculum theory should be operative.

It is to be hoped, therefore, that leaders in the current efforts to reform the mathematics curriculum will seize this opportunity to make their own philosophy of education explicit, and will also see to it that the psychological principles underlying their proposals are also clearly enunciated. They should take every possible step to insure that the emerging curriculum will not only be modern and mathematically sound, but also teachable. Above all, it must be *learnable* by those for whom it is designed. The intellectual demands of mathematics are such that the subject inevitably acts as a selector. Unfortunate decisions concerning what is teachable may make the subject overly selective in spite of demands for more rather than fewer minds familiar with its discipline. Mathematicians should not attempt to make these decisions alone.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Membership Committee

Mary C. Rogers, Chairman, Membership Committee, Roosevelt Junior High School, Westfield, New Jersey

The Membership Committee of the National Council of Teachers of Mathematics is pleased to report to you the results of the 1954–1956 Membership Campaign, and to thank you for your fine support and direct helpfulness in bringing about the generous increase in membership which was accomplished. It was this assistance on the part of each of you as individuals which has largely made this accomplishment possible. We are indeed grateful to you for your enthusiastic cooperation.

Record of membership growth

We are submitting herewith the latest membership analysis based on the membership count by states received from the Washington Office early in May 1956. Please study it carefully for the over-all picture it presents, and for the results which you have helped bring about in your various states.

In this analysis, you will notice a total count of 13,827 members. This represents a relative achievement of 92.2% of the total 15,000, the goal set for us in April 1954.

More specifically, we are pleased to announce:

1. 56% of all states and regions show a relative achievement of 90–188% of their established goals.
2. 13% have attained from 86–89% relative achievement.
3. 12% have a standing of 80–84% of goal.
4. 13 states have "gone over the top," while 14 states have fewer than 20 members each to secure to reach their goals.
5. The following 13 states and territories have reached their goals or gone beyond them:

U. S. Possessions	188%	Canada	116%
Arizona	177%	New Hampshire	114%
Nevada	164%	Wisconsin	111%
South Dakota	145%	California	101%
Oregon	131%	Texas	100%
Utah	125%	Vermont	100%
District of Columbia	116%		

6. Membership achievement in these 16 states shows 90–99% of established goals:

Florida	99%	Colorado	93%
Washington	99%	Delaware	93%
Montana	98%	Kansas	92%
New York	98%	Pennsylvania	92%
Maine	97%	Iowa	91%
Wyoming	97%	Michigan	91%
Connecticut	96%	New Jersey	91%
Maryland	94%	Illinois	90%

7. Membership in the following 7 states shows 85–89% achievement:

Massachusetts	89%	Virginia	87%
New Mexico	89%	Louisiana	86%
Oklahoma	89%	Mississippi	86%
Indiana	88%		

Other indications of progress include:

Leaders in Membership Totals

1. New York	1152	6. Ohio	613
2. Illinois	1075	7. Michigan	544
3. California	888	8. New Jersey	511
4. Pennsylvania	844	9. Indiana	478
5. Texas	673	10. Wisconsin	465

Leaders in Membership Growth
(Since April 1955)

1. Illinois	311	6. Wisconsin	158
2. California	309	7. New Jersey	145
3. New York	289	8. Michigan	144
4. Texas	193	9. Ohio	133
5. Pennsylvania	189	10. Indiana	126

The National Council of Teachers of Mathematics
Analysis of Membership Growth, April 1955-May 1956

	April 1955	*April 1956*	*May 1956*	*Goals*	*Per Cents*
Alabama	108	132	135	170	79%
Arizona	40	104	106	60	177%
Arkansas	92	102	106	171	62%
California	579	873	888	879	101%
Colorado	143	168	168	180	93%
Connecticut	180	219	220	228	96%
Delaware	45	67	66	71	93%
District of Columbia	131	213	214	185	116%
Florida	237	337	346	351	99%
Georgia	145	150	152	186	82%
Idaho	12	12	12	18	67%
Illinois	764	1,018	1,075	1,188	90%
Indiana	352	477	478	542	88%
Iowa	202	264	270	296	91%
Kansas	217	276	285	311	92%
Kentucky	73	98	98	123	80%
Louisiana	176	235	236	276	86%
Maine	52	71	67	69	97%
Maryland	207	262	269	285	94%
Massachusetts	323	381	389	440	89%
Michigan	400	571	544	600	91%
Minnesota	246	308	311	391	80%
Mississippi	83	103	106	123	86%
Missouri	238	278	278	330	84%
Montana	50	59	59	60	98%
Nebraska	105	127	126	165	76%
Nevada	17	23	23	14	164%
New Hampshire	44	67	74	65	114%
New Jersey	366	490	511	561	91%
New Mexico	61	78	79	89	89%
New York	863	1,135	1,152	1,181	98%
North Carolina	172	198	198	255	78%
North Dakota	31	27	28	44	64%
Ohio	480	596	613	740	83%
Oklahoma	163	209	212	237	89%
Oregon	109	165	176	134	131%
Pennsylvania	655	813	844	920	92%
Rhode Island	50	51	52	71	73%
South Carolina	83	100	102	137	74%
South Dakota	33	53	52	36	145%
Tennessee	154	187	207	246	84%
Texas	480	653	673	672	100%
Utah	43	60	60	48	125%
Vermont	25	41	41	41	100%
Virginia	250	309	310	357	87%
Washington	167	226	226	228	99%
West Virginia	77	77	77	188	41%
Wisconsin	307	415	465	420	111%
Wyoming	27	34	35	36	97%
TOTALS	9,857	12,912	13,214	14,388	92%
Hawaii		37	37		
and other U. S. Possessions	43	102	102	74	188%
Canada	142	195	233	201	116%
Foreign	214	222	241	339	71%
GRAND TOTALS	10,256	13,468	13,827	15,002	92%

GREATEST RELATIVE GROWTH
(Since April 1955)

1. U. S. Possessions	9. Wisconsin
2. Arizona	10. Delaware
3. New Hampshire	11. Florida
4. Canada	12. Illinois
5. District of Columbia	13. New Jersey
6. Oregon	14. Texas
7. South Dakota	15. Utah
8. California	

PLANS FOR THE FUTURE

This record of membership progress was reported to the National Council Board of Directors at its 1956 summer meeting in Los Angeles. Your membership committee was asked to continue its work with you for another two-year period. Our goal for total membership remains 15,000; suggested goals for states and regions will remain as heretofore established.

Most commendable growth in NCTM membership has been made, but the services of the Council still are not reaching a great many mathematics teachers and other persons throughout the country who are interested in the betterment of mathematics education. We want very much to correct this inadequacy, and we shall continue to work hard to do so. With your increasingly enthusiastic support and assistance we are confident we shall make great strides toward reaching these people.

How can you help membership growth in your state? How can you further NCTM membership progress toward the 15,000 goal? May we suggest that future procedures follow closely those you have found effective in the past.

1. Will each of you please secure at least *one new member* for the Council? We have found the "Each-One-Win-One" technique to be one of our most valuable aids and urge its continuance and positive use by *all current leaders and members.*
2. *Prompt renewals* of present memberships greatly facilitate the keeping of records and the preparation of frequent reports.
3. You who are members of the mathematics staff in colleges of education and similar education centers can be of invaluable assistance through *your continued stimulation of interest* in NCTM services among your students.
4. A similar service can be rendered with increasingly fine results by you who are *supervisors or department chairmen* in your work with your teachers.
5. *Many state and other local associations* of mathematics teachers are affording the Council most valuable publicity at your meetings and through your publications. A continuation and expansion of this fine support will be deeply appreciated.
6. Your continued strong support of NCTM *state representatives* will greatly facilitate the outstanding work they are performing for the Council.
7. May we remind you that library and other institutional subscriptions are counted toward the 15,000 goal, although they are not considered as memberships.

Your membership committee will keep you informed of membership progress as often as it is available to us. You will hear from us at least twice a year through this publication. Reports will be made at all National Council meetings and will be available for release at your local meetings and through your local publications.

We welcome your advice and suggestions at any time for the improvement of this membership service.

Please accept our best wishes for a most successful year in all of your professional endeavors.

Seventeenth Christmas Meeting

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Arkansas State College, Jonesboro, Arkansas

December 27–29, 1956

Your very able President, *Howard F. Fehr*, will preside at the first general session.

Mrs. Marie Wilcox, your charming Past-President, will address us at a general session on Friday morning, December 28. The title of her talk: "Now is the Time." Mrs. Wilcox's presentation will prove both inspirational and informational.

Henry Van Engen, the Editor of THE MATHEMATICS TEACHER, will participate in discussions of pertinent problems in the teaching of mathematics with us.

Report from Mr. Frank Allen, Chairman of The Curriculum Committee of The National Council of Teachers of Mathematics.—Mr. Allen's committee of seven very able people are getting things done! The committee's final report, in my opinion, will be as significant as the 1923 report, *The Reorganization of Mathematics in Secondary Education*, or the 15th yearbook, *The Place of Mathematics in Secondary Education.* Plan to come to Jonesboro to hear what Mr. Allen has to say.

Beberman and Page make "Time"—More than half the Education section of *Time* dated July 23, 1956, was devoted to the type of mathematics program being developed in Illinois. Mr. Beberman and participants will tell us about the latest developments in their experimental high school mathematics program. The secondary mathematics program is due for change. Be in Jonesboro during the Christmas holidays to hear what Mr. Beberman and participants will have to propose to improve the mathematics program.

Much is being said about modern mathematics. We shall have outstanding mathematicians tell us what they believe the secondary schools can do to incorporate new ideas into present programs.

Leaders in the elementary field will be in Jonesboro to inform and inspire us. *Laura H. Eads* from New York, *Louis E. Ulrich* from Milwaukee, Wisconsin, *Glenadine Gibb* from Cedar Falls, Iowa and many others will be participants in the Christmas meeting.

Are you up-to-date on the Science Teaching Improvement Program? Come to Jonesboro to hear experts tell us about it.

Thomas J. Fritts, Exhibits Manager, Interpreter, U. S. Atomic Energy Commission, Oak Ridge, Tennessee, will speak on this topic: "Aspects of Nuclear Science as Related to Peacetime Uses." Rather an inviting topic, isn't it?

There are many, many more highlights too numerous to mention. May I mention only a few more of the outstanding leaders in the field of the teaching of mathematics, who will be on the program: *Edith Woolsey, Sheldon Myers, Philip Peak, Carl H. Shuster, Clifford Bell, Lynwood Wren, Charles W. Curtis, W. A. Gager*, and *Dale Carpenter.*

Who is the banquet speaker? *B. J. Newchurch*, Assistant Director, Esso Research Laboratories, Baton Rouge, Louisiana. What is his topic? "Mathematics in Industry."

You will have an opportunity to hear that dynamic speaker, *Phillip S. Jones*, from the University of Michigan. His

topic: "Enriching the Elementary Mathematics Courses for Engineers."

Oh yes—Reports on those important Institutes being held all over the United States.

Our friend and local Chairman, *Lyle Dixon*, has arranged several sight-seeing and educational tours. He is also planning for lots of fun.

We shall be looking for you in Jonesboro during the Christmas holidays. Make plans now.

Milton W. Beckmann, Program Chairman

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

CHRISTMAS MEETING

December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

ANNUAL MEETING

March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

JOINT MEETING WITH NEA AND NSTDA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

Other professional dates

Chicago Elementary Teachers Mathematics Club
December 10, 1956
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Genevieve E. Johnson, Volta School, Chicago, Illinois

Women's Mathematics Club of Chicago and Vicinity
February 9, 1957
Tearoom of The Fair Store, State and Adams Streets, Chicago, Illinois
Ruth Woerner, 11715 S. 82nd Court, Palos Park, Illinois

THE MATHEMATICS TEACHER

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EDITORIAL OFFICE

Henry Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

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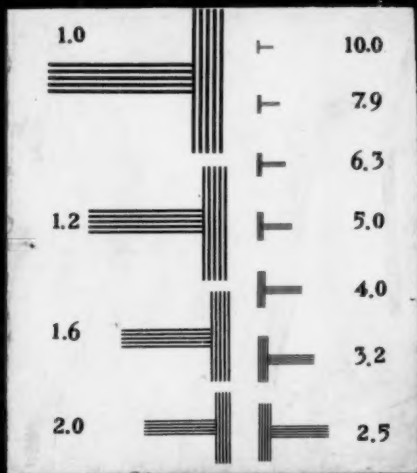
RESOLUTION CHART



100 MILLIMETERS

INSTRUCTIONS Resolution is expressed in terms of the lines per millimeter recorded by a particular film under specified conditions. Numerals in chart indicate the number of lines per millimeter in adjacent "T-shaped" groupings.

In microfilming, it is necessary to determine the reduction ratio and multiply the number of lines in the chart by this value to find the number of lines recorded by the film. As an aid in determining the reduction ratio, the line above is 100 millimeters in length. Measuring this line in the film image and dividing the length into 100 gives the reduction ratio. Example: the line is 20 mm. long in the film image, and 100/20 = 5.

Examine "T-shaped" line groupings in the film with microscope, and note the number adjacent to finest lines recorded sharply and distinctly. Multiply this number by the reduction factor to obtain resolving power in lines per millimeter. Example: 7.9 group of lines is clearly recorded while lines in the 10.0 group are not distinctly separated. Reduction ratio is 5, and 7.9 x 5 = 39.5 lines per millimeter recorded satisfactorily. 10.0 x 5 = 50 lines per millimeter which are not recorded satisfactorily. Under the particular conditions, maximum resolution is between 39.5 and 50 lines per millimeter.

Resolution, as measured on the film, is a test of the entire photographic system, including lens, exposure, processing, and other factors. These rarely utilize maximum resolution of the film. Vibrations during exposure, lack of critical focus, and exposures yielding very dense negatives are to be avoided.